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Negative eddy viscosity in isotropically forced two-dimensional flow: linear and nonlinear dynamics

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The existence of two-dimensional flows with an isotropic and negative eddy viscosity is demonstrated. Such flows, when subject to a very weak large-scale perturbation of wavenumber k will amplify it with a rate proportional to k^2, independent of the direction.

Specifically, it is assumed that the basic (unperturbed) flow is space–time periodic, possesses a centre of symmetry (parity-invariance) and has three- or six-fold rotational invariance to ensure isotropy of the eddy-viscosity tensor.

The eddy viscosities emerging from the multiscale analysis are calculated by two different methods. First, there is an expansion in powers of the Reynolds number which can be carried out to large orders, and then extended analytically (thanks to a meromorphy property) beyond the disk of convergence. Secondly, there is a spectral method. The two methods typically agree within a fraction of 1%.

A simple example, the 'decorated hexagonal flow', of a time-independent flow with isotropic negative eddy viscosity is given. Flows with randomly chosen Fourier components and the required symmetry have typically a 30% chance of developing a negative eddy viscosity when the Reynolds number is increased.

For basic flow driven by a prescribed external force and sufficiently strong large-scale flow, the analysis is extended to the nonlinear régime. It is found that the large-scale dynamics is governed by a Navier–Stokes or a Navier–Stokes–Kuramoto–Sivashinsky equation, depending on the sign and strength of the eddy viscosity. When the driving force is not mirror-symmetric, a new 'chiral' nonlinearity appears. In special cases, the large-scale equation reduces to the Burgers equation. With chiral forcing, circular vortex patches are strongly enhanced or attenuated, depending on their cyclonicity.

1. Introduction

Eddy-transport coefficients, as used here, characterize the way a given basic flow, spatially periodic (cellular flow), responds to a weak large-scale perturbation.

A well-known instance is the addition of a *passive scalar* advected by the basic flow and subject to molecular diffusion. If the basic flow has enough symmetry (e.g. square symmetry in two dimensions), the large-scale behaviour of the cell-averaged scalar concentration is governed by a diffusion equation. The eddy-diffusivity is positive; otherwise, the maximum principle which holds for any advection–diffusion equation would be violated.

The concept of eddy viscosity arises when the large-scale perturbation affects the distribution of momentum and vorticity within the basic flow. The governing equation is the linearized Navier–Stokes equation. Because of the interplay of momentum and vorticity perturbations, the latter never reduces to an advection–diffusion equation, even in two dimensions where the vorticity is a scalar. As a consequence, a large-scale momentum gradient may produce a cell-averaged momentum flux which is in the same direction as the gradient, rather than opposite to it. Such flow displays a large-scale instability with growth rate proportional to the square of the wavenumber. It is traditional to refer to such flow as having *negative eddy viscosity* (Starr 1968; Kraichnan 1976). The best known example is the two-dimensional Kolmogorov flow with stream function $\Psi = \cos x_1$. Its eddy viscosity for large-scale perturbations transverse to the basic flow is given by

$$\nu_E = \nu - 1/(2\nu), \tag{1}$$

which becomes negative when $\nu < 2^{-\frac{1}{2}}$ (Meshalkin & Sinai 1961; Nepomnyashchy 1976; Sivashinsky 1985; Dubrulle & Frisch 1991; Hénon & Scholl 1991).

Negative eddy viscosity has been invoked many times as a possible explanation of common instabilities in astrophysical and geophysical flow, among them the differential rotation of the Sun (see e.g. Rüdiger 1989). Note that, in general, the eddy viscosity is a fourth-order tensor, as needed to linearly relate the large scale velocity gradient and the ensuing momentum flux. Kraichnan (1976) derived the expression for the eddy viscosity within a closure framework and obtained negative values in two dimensions. He then used this result to interpret the reverse flow of energy in the inverse cascade of two-dimensional turbulence.

As is now known, there is no need to resort to closure to calculate eddy viscosities, since the latter can be calculated by multiscale techniques, developed by Nepomnyashchy (1976), Sivashinsky (1985), and others for the two-dimensional case and extended by Dubrulle & Frisch (1991, referred to herein as DF) to higher dimensions. Such methods indeed give the exact value of the eddy-transport coefficients. However, they require the solution of auxiliary problems having analytical solutions only in special cases, for example when the basic flow is layered, i.e. depending on a single space-coordinate (DF). This is why the only flows known for certain to possess a negative eddy viscosity are highly anisotropic.

It was an open problem if two-dimensional flows exist having an *isotropic* eddy viscosity which is negative. This question was recently addressed by Sivashinsky & Frenkel (1992) who considered low-Reynolds-number flow and showed that the first correction to the molecular viscosity (in powers of the Reynolds number) can be negative when the basic flow is time-dependent. (DF had shown that this is ruled out for time-independent flow.) As Sivashinsky & Frenkel (1992) pointed out, one cannot be certain that, at some finite Reynolds numbers, the reduction of the eddy viscosity may become prominent enough to make it negative.

In a recent note (Vergassola, Gama & Frisch 1993), we have briefly demonstrated the existence of at least one deterministic time-independent, space-periodic flow, which has an isotropic eddy viscosity above some critical Reynolds number. The present paper has a considerably broader scope and is organized as follows. In §2 we formulate our problem and discuss the basic symmetries (parity, isotropy, chirality) which will be of importance for our studies. In §3 we present the multiscale technique needed to derive the eddy viscosity. Section 4 is devoted to two general methods for calculating the eddy-viscosity tensor of a given flow. First, in §4.1, we present a method based on the analytic continuation by Padé approximants of series giving the eddy viscosity in powers of the Reynolds number. Then, in §4.2, we present a numerical strategy based on (pseudo-) spectral numerical solutions of the auxiliary problems generated in the multiscale expansion. In §§5–7, we specialize to flows having enough rotational symmetry to make the eddy viscosity isotropic. We first study a time-independent flow with a single wavenumber, closely related to the one considered by Sivashinsky & Yakhot (1984), the eddy viscosity of which is shown to be always positive (§5.1). We then turn to a three-wavenumber flow, the 'decorated hexagonal flow', which is the first known instance of isotropic negative eddy viscosity (§5.2). In §6, we address the question of how common the phenomenon of negative eddy viscosity is when the basic flow has randomly selected Fourier components. Section 7 is devoted to time-dependent basic flows. We first give finite-Reynolds-numbers results for the deterministic Sivashinsky & Frenkel (1992) flow. Then, in §7.1, we consider the case of random flows which are δ-correlated in time.

In the next two Sections we address the question of the nonlinear dynamics of large-scale flow, a problem that is of particular relevance when there is an instability of negative eddy viscosity. For the linear theory, it was sufficient to prescribe the basic flow subject to the large-scale perturbation. For the nonlinear theory, we must specify how the basic flow is maintained. The simplest way to maintain cellular flow is through a prescribed external force which is a function of space and time, independent of the velocity.

Experimentally, two-dimensional flow subject to such forcing can be realized by a spatially periodic varying magnetic field acting on a thin layer of an electrolyte (Bondarenko, Gak & Dolzhansky 1979). Alternatively, a uniform magnetic field can be used in combination with an array of current-injecting electrodes to produce a wide range of spatial and temporal modulations, as shown by Sommeria (1986) who also checked quantitatively that the governing equation is the two-dimensional Navier–Stokes equations with an additional friction term proportional to the velocity. The latter stems from bottom friction. Sommeria showed that the friction term can be made sufficiently small to affect only the very largest scales by using appropriate experimental parameters. As a consequence, features such as flow symmetry and negative eddy viscosity are mostly unaffected.

The Sections devoted to the nonlinear large-scale dynamics are organized as follows. First, in §8, we consider forcing possessing mirror-symmetry (non-chiral) and handle successively the cases of positive eddy viscosity (§8.1) and marginally negative eddy viscosity (§8.2). The latter, leads to the Navier–Stokes–Kuramoto–Sivashinsky equation, the properties of which will be recalled briefly. Indeed, it has been investigated numerically in detail by Gama, Frisch & Scholl (1991) at a time when it was only conjectured that there exists flow with isotropic and negative eddy viscosity. Chiral forcing, already briefly discussed by Vergassola (1993), leads to surprisingly novel large-scale dynamics discussed in §9.

Concluding remarks are presented in §10. In the main body of the paper the emphasis is on concepts and results. Technical details are presented in five Appendices.

2. Formulation and symmetries

In this paper we are investigating two-dimensional incompressible flow subject to external forcing. The general D-dimensional case was considered in DF. In the two-dimensional case, the formalism can be made more compact, using a stream-function representation as in Nepomnyashchy (1976), Sivashinsky & Yakhot (1984) and Sivashinsky & Frenkel (1992). The velocity field $\boldsymbol{u} = (u_i)$ is written as

$$u_i = \varepsilon_{ij}\,\partial_j\,\Psi, \quad i = 1,2. \tag{2}$$

Here, ε_{ij} is the fundamental antisymmetric tensor ($\varepsilon_{12} = -\varepsilon_{21} = 1$, zero otherwise) and ∂_j stands for $\partial/\partial x_j$.

The Navier–Stokes equation for the *basic flow* reads

$$\partial_t\,\partial^2\Psi + J(\partial^2\Psi, \Psi) = \nu\,\partial^2\,\partial^2\Psi - \varepsilon_{ij}\,\partial_i\,f_i. \tag{3}$$

Here, ∂^2 is the Laplacian, $J(f,g) = \varepsilon_{ij}(\partial_i f)(\partial_j g)$ is the Jacobian, ν is the (kinematic molecular) viscosity and $f = (f_i)$ is the external force.

We shall assume that the basic flow Ψ is a prescribed function, periodic in x_1, x_2 and t. The periods in x_1 and x_2 need not be the same. The force f is chosen in such a way as to balance (3) and thus will be a function of the viscosity, when the latter is varied. Whether the basic flow or the external force is prescribed makes no difference in the linear theory, but matters in the nonlinear case. Studies of the Kolmogorov flow usually assume that the basic flow rather than the force is prescribed, as we shall here assume. One thereby avoids questions about non-uniqueness of the basic flow, questions which are of interest, but are only weakly connected with the main thrust of this paper. We shall also assume that the basic flow is stable with respect to small perturbations having the same spatial periodicity since, again, small-scale instabilities are not our concern. This condition is satisfied when the Reynolds number for the basic flow is small enough. The latter is defined as

$$R = \langle \Psi^2 \rangle^{\frac{1}{2}}/\nu. \tag{4}$$

(Note that Ψ has the dimension [length] × [velocity].)

We also assume that

$$\langle \boldsymbol{u} \rangle = \langle f \rangle = 0. \tag{5}$$

Here, the mean value $\langle \cdot \rangle$ denotes the average over the space and time periodicities. Condition (5) is no real restriction, since it can be satisfied by performing a suitable Galilean transformation.

The restrictive hypothesis of periodicity is made mostly for simplicity. The case of random homogeneous stationary flow with rapidly decreasing correlations (mixing) can be handled in principle by the same method with minor modifications at the formal level (angular brackets being then reinterpreted as ensemble averages). An example will be found in §7.1.

We now formulate the problem of the large-scale perturbation. Let us replace in the Navier–Stokes equation (3) the basic flow Ψ by $\Psi + \psi$, where ψ, the *large-scale perturbation*, is not restricted to having any spatial periodicity. The perturbation ψ is taken to be 'small'. How small exactly is a question we shall return to at length. Since Ψ satisfies the Navier–Stokes equation, ψ satisfies an equation in which the force drops out:

$$\partial_t\,\partial^2\psi + J(\partial^2\Psi, \psi) + J(\partial^2\psi, \Psi) + J(\partial^2\psi, \psi) = \nu\,\partial^2\,\partial^2\psi. \tag{6}$$

When the large-scale perturbation is sufficiently weak, the nonlinear term $J(\partial^2\psi, \psi)$ is negligible and we simply obtain for ψ the linearized Navier–Stokes equation. Sections 3–7 will deal with the linear theory, for which it is irrelevant how the basic flow is maintained.

Let us denote by $\psi^{(0)}$ the large-scale component of the perturbation. (Precise definitions justifying this notation will be given later.) A gradient expansion in the spirit of Moffatt (1974) indicates that the large-scale equation will have the form

$$\partial_t\,\partial^2\psi^{(0)} = -\alpha_{ijl}\,\partial_i\,\partial_j\,\partial_l\,\psi^{(0)} + \nu_{ijlm}\,\partial_i\,\partial_j\,\partial_l\,\partial_m\,\psi^{(0)} + \ldots. \tag{7}$$

The third-order tensor α_{ijl} is called the AKA tensor (Frisch, She & Sulem 1987*b*; Sulem *et al.* 1989). It vanishes when the basic flow is *parity-invariant*, i.e. has a centre of symmetry. If the latter is taken as origin, this means that the basic stream function Ψ satisfies

$$\Psi(-\boldsymbol{x},t) = \Psi(\boldsymbol{x},t). \tag{8}$$

Here, parity-invariance will be assumed because it is the most general condition which guarantees the absence of AKA-type tensors. In two dimensions the AKA-effect can only lead to dispersion (an immediate consequence of (7)) and is thus less interesting than in three dimensions where instabilities are permitted.

In DF it was shown that the large-scale dynamics for weak perturbations of parity-invariant flow is indeed of the form (7), with no α_{ijl} terms. The most general anisotropic case contains a fourth-order eddy-viscosity tensor ν_{ijlm} (Krause & Rüdiger 1974). Since the ν_{ijlm} tensor is contracted with four ∂'s, it may without loss of generality be completely symmetrized in its four indices. If the basic flow is random and isotropic a scalar eddy viscosity will of course be obtained, i.e.

$$\nu_{ijlm} = \nu_E\, \delta_{ij}\, \delta_{lm}. \tag{9}$$

The coefficient ν_E is then simply called the 'eddy viscosity'. Note that we do include the molecular contribution in our definition of ν_E.

Deterministic flow, as used here, cannot be invariant under rotations (unless it vanishes). Still, it is well known in crystallography (Landau & Lifshitz 1970) and in the theory of lattice gases (Frisch, Hasslacher & Pomeau 1986*a*; Frisch *et al.* 1987*a*) that six-fold (sixty degree) rotational invariance ensures isotropy of certain tensors up to the fourth order. Such tensors have to be at least symmetric under exchanges of two pairs of indices. For completely symmetrical tensors, three-fold rotational symmetry is enough for isotropy. This is why there will be considerable emphasis on such symmetry in this paper.

Another symmetry which will turn out to be important in the nonlinear theory is *mirror symmetry*, i.e. symmetry with respect to an axis. Taking this axis to be in the x_2 direction, mirror symmetry is expressed as

$$\Psi(-x_1, x_2, t) = -\Psi(x_1, x_2, t). \tag{10}$$

For an example of a mirror-symmetric flow which is not parity-invariant (no centre of symmetry), see figure 5. For an example of a parity-invariant flow which is chiral (no axis of symmetry), see figure 3. It is easily checked that mirror-symmetry is *dynamical*, i.e. compatible with the Navier–Stokes equation in the absence of a force, or when the force itself is mirror-symmetric. Flows which do not possess mirror symmetry are said to be *chiral*. As we shall see, this matters only for the nonlinear theory.

Finally, it will be useful to consider flow with *symmetric streamlines*, such that

$$\Psi(-x, y) = \Psi(x, y); \tag{11}$$

This is, however, not a dynamical symmetry.

3. Multiscale technique

The general technique for finding eddy-viscosity tensors is the so-called multiscale technique, also known as *homogenization* (Bensoussan, Lions & Papanicolaou 1978). This technique as well as the heuristic interpretation of the appropriate scalings are presented in detail in DF. Here, we shall emphasize only the aspects that are particular to the two-dimensional case, when using stream functions. The starting point is the linearized Navier–Stokes equation

$$\mathscr{A}\psi \equiv \partial_t \partial^2\psi + J(\partial^2\psi, \Psi) + J(\partial^2\Psi, \psi) - \nu\,\partial^2\partial^2\psi = 0. \tag{12}$$

We are interested in the dynamics on *large*-scales assumed to be $O(\epsilon^{-1})$; the associated (diffusive) timescale is then $O(\epsilon^{-2})$. The space–time variables of the basic flow, the so-called periodic *fast variables*, are denoted $\boldsymbol{x}$ and t. In addition, we introduce slow variables, viz. $\boldsymbol{X} = \epsilon\boldsymbol{x}$ and $T = \epsilon^2 t$. As usual, the multiscale expansion pretends that fast variables and slow variables are independent. It follows that

$$\partial_i \to \partial_i + \epsilon\nabla_i, \quad \partial_t \to \partial_t + \epsilon^2\,\partial_T, \tag{13}$$

where we denote the derivatives with respect to fast space variables by the symbol ∂ and those with respect to slow space variables by ∇. The solution $\psi(\boldsymbol{x}, t; \boldsymbol{X}, T)$ is sought as a series in ϵ:

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + \dots, \tag{14}$$

where all the functions $\psi^{(n)}$ depend *a priori* on both fast and slow variables. Use of the expansion (14) and the rule of decomposition of derivatives (13) in (12) gives a hierarchy of equations which are derived in detail in Appendix A. Here, we just make some general remarks. All the equations involve the operator A, which is the same as $\mathscr{A}$ of (12), but restricted to periodic functions (fast variables). Because of the presence of fast derivatives on the right in every term of A, constant functions are in its null-space. As is usual in singular perturbation theory, A is not invertible. Still, its restriction to functions of zero mean value, denoted $\tilde{\mathrm{A}}$ as in DF, is invertible at small enough Reynolds numbers. Indeed, $\tilde{\mathrm{A}}$ is then close to the bi-Laplacian $\nu\partial^2\partial^2$, which is invertible on functions of zero mean. We must of course limit our investigation to Reynolds numbers such that no eigenvalue of $\tilde{\mathrm{A}}$ crosses the imaginary axis, since this would imply a small-scale instability. We now write the first three members of the hierarchy of equations corresponding to $O(\epsilon^0)$, $O(\epsilon^1)$ and $O(\epsilon^2)$, respectively:

$$\mathrm{A}\psi^{(0)} = 0, \tag{15}$$

$$\mathrm{A}\psi^{(1)} = \varepsilon_{\alpha i}(\partial_i\partial^2\Psi)\nabla_\alpha\psi^{(0)}, \tag{16}$$

$$\mathrm{A}\psi^{(2)} = [-2\partial_t\partial_\alpha - \varepsilon_{i\alpha}(\partial_i\partial^2\Psi) + 2\varepsilon_{ij}(\partial_i\Psi)\partial_j\partial_\alpha + \varepsilon_{i\alpha}(\partial_i\Psi)\partial^2 + 4\nu\,\partial_\alpha\partial^2]\nabla_\alpha\psi^{(1)}. \tag{17}$$

Here, and below, it is understood that partial derivative operators (in fast or slow variables) such as ∂ and ∇ act on anything to the right unless immediately preceded by an open parenthesis, in which case they act only within the corresponding parenthetical group. There is no difference between Greek and Roman indices. All three of the above equations, as well as all subsequent equations in the hierarchy are of the form

$$\mathrm{A}f = g. \tag{18}$$

The solvability condition is then $\langle g\rangle = 0$. It is easily checked that it is satisfied for all three of the above equations. This situation is a bit different from the one presented in DF. When using a velocity formalism, as in DF, the solvability for the second-level equation gives the absence of the AKA-effect, while the solvability of the third-level equation gives the eddy viscosity tensor. When using stream functions, the first non-trivial solvability condition (absence of AKA-effect) appears at the third level and the eddy viscosity at the fourth.

A quick way of obtaining all the solvability conditions is given in Appendix A. To explicitly write the solvability conditions up to $O(\epsilon^4)$, one needs only the solutions of the first three equations (15)–(17). The first one is trivial: $\psi^{(0)}$ will be in the null-space of A, thus independent of the fast variables. The second and third equations (16) and (17), may be solved for $\psi^{(1)}$ and $\psi^{(2)}$ in the restricted class of functions of zero mean, since additive constants will not affect subsequent solvability conditions. Since (16) is linear and its right-hand side involves the slow variables only through the factor $\nabla_\alpha \psi^{(0)}$, it follows that

$$\psi^{(1)} = \boldsymbol{Q}\cdot\nabla\psi^{(0)} + \langle\psi^{(1)}\rangle, \tag{19}$$

where $\boldsymbol{Q}$ is a suitable vector depending only on the fast space variables. Substituting this into (17), we see that the solution of the latter may be written as

$$\psi^{(2)} = S_{\alpha\beta}\nabla_\alpha\nabla_\beta\psi^{(0)} + \boldsymbol{Q}\cdot\nabla\langle\psi^{(1)}\rangle + \langle\psi^{(2)}\rangle, \tag{20}$$

where again the tensor $S_{\alpha\beta}$ depends only on fast variables.

Finally, the equation for the large-scale perturbation $\langle\psi^{(0)}\rangle$ stemming from the solvability of the fourth-order equation is

$$\partial_T \nabla^2\psi^{(0)} = \nu_{\alpha\beta\gamma\eta}\nabla_\alpha\nabla_\beta\nabla_\gamma\nabla_\eta\psi^{(0)}, \tag{21}$$

where the eddy-viscosity tensor is given by

$$\nu_{\alpha\beta\gamma\eta} = \nu\delta_{\alpha\beta}\delta_{\gamma\eta} - \varepsilon_{\alpha i}\delta_{\beta\gamma}\langle Q_\eta \partial_i \Psi\rangle - 2\varepsilon_{\alpha i}\langle(\partial_\beta S_{\gamma\eta})(\partial_i \Psi)\rangle. \tag{22}$$

The only non-trivial problem is to calculate Q_α and $S_{\alpha\beta}$. This requires the solution of linear partial differential equations with periodic coefficients.

4. Determination of the eddy viscosity

As is standard in homogenization methods, the determination of the transport coefficients requires the solution of auxiliary problems formulated solely in terms of the fast variables. To calculate the eddy-viscosity tensor, given by (22), we need to know the vector $Q_\alpha(x,t)$ and the tensor $S_{\alpha\beta}(x,t)$, which depend only on fast variables and satisfy

$$\tilde{\mathrm{A}}Q_\alpha = \varepsilon_{\alpha i}(\partial_i\partial^2\Psi), \tag{23}$$

$$\begin{aligned}\tilde{\mathrm{A}}S_{\alpha\beta} = &-2\partial_t(\partial_\alpha Q_\beta) - \varepsilon_{i\alpha}(\partial_i\partial^2\Psi)Q_\beta \\ &+ 2\varepsilon_{ij}(\partial_i\Psi)(\partial_j\partial_\alpha Q_\beta) + \varepsilon_{i\alpha}(\partial_i\Psi)(\partial^2 Q_\beta) + 4\nu(\partial_\alpha\partial^2 Q_\beta),\end{aligned} \tag{24}$$

where
$$\tilde{\mathrm{A}} = \partial_t\partial^2 + J(\partial^2\bullet,\Psi) + J(\partial^2\Psi,\bullet) - \nu\partial^2\partial^2, \tag{25}$$

is the linearized Navier–Stokes operator around the basic flow Ψ restricted to periodic functions of zero mean. Equations (23) and (24) follow immediately from (16), (17), (19) and (20). When the basic flow is time-independent, the first term $\partial_t\partial^2$ in the operator $\tilde{\mathrm{A}}$ may be omitted. The assumption we made ensures that (23) and (24) have unique solutions, which are bounded space–time periodic functions with zero mean, as long as the Reynolds number is small enough to prevent instabilities at small scales.

We shall now describe two very different strategies for solving the auxiliary problems (23) and (24).

4.1. *Analytic continuation in the Reynolds number*

Our first method, which is mostly analytic, will be described here only for the case of *time-independent* basic flow, although it is easily extended to include time-dependence. It follows from (22), (23) and (24) that the only singularities of the eddy-viscosity tensor, when the viscosity is varied, originate from lack of invertibility of the linearized Navier–Stokes operator $\tilde{\mathrm{A}}$. Indeed, quantities such as Q_α and $S_{\alpha\beta}$ are obtained by solving equations of the form $\tilde{\mathrm{A}}f = g$. An important consequence of the fact that these linearized Navier–Stokes operators are formulated on the bounded domain of periodicity is that the eddy viscosity is a meromorphic function of the Reynolds number $R \propto 1/\nu$, when the latter is extended to complex values. In other words, its only singularities at finite distance are poles. The proof is very simple. We apply to (18) the inverse of the bi-Laplacian $\partial^{-2}\partial^{-2}$ (which is well-defined on functions of zero mean) and use (12) to obtain

$$(\mathbf{B}-\nu I)f = \partial^{-2}\partial^{-2}g, \tag{26}$$

where I is the identity and

$$\mathbf{B}f = \partial^{-2}\partial^{-2}\{J(\partial^2 f, \Psi)+J(\partial^2\Psi, f)\}. \tag{27}$$

Hence, f can become singular only when ν is an eigenvalue of $\mathbf{B}$. It is easily checked that $\mathbf{B}$ is a compact operator in the space of square-integrable periodic functions because it has more integrations than differentiations. The meromorphy of the eddy viscosity in the inverse viscosity is then a consequence of a classical theorem on the spectrum of compact operators (Dunford & Schwartz 1966).

We now observe that solution of (26) has a straightforward series expansion in powers of $1/\nu$, namely

$$f = f_1\nu^{-1}+f_2\nu^{-2}+\ldots+f_n\nu^{-n}+\ldots, \tag{28}$$

where $f_1 = -\partial^{-2}\partial^{-2}g$, and

$$f_{n+1} = \partial^{-2}\partial^{-2}[J(\partial^2 f_n, \Psi)+J(\partial^2\Psi, f_n)], \quad n \geqslant 1. \tag{29}$$

In (28) there is no f_0 term since the mean of f vanishes. When g is itself a series in $1/\nu$, the recursion relation for the f_n is an obvious generalization of (29). To calculate Q_α, the function g is taken to be the right-hand side of (23). To calculate $S_{\alpha\beta}$, the function g is taken to be the right-hand side of (24), which must itself be expanded in powers of $1/\nu$. After substitution of (28) in (22), we obtain an expansion for the eddy-viscosity tensor

$$\nu_{\alpha\beta\gamma\eta} = \nu\delta_{\alpha\beta}\delta_{\gamma\eta}+\sum_{n=1}^{\infty}\nu^{(n)}_{\alpha\beta\gamma\eta}\nu^{-n}. \tag{30}$$

The expansion is guaranteed to converge for small enough Reynolds numbers (large enough ν). The radius of convergence is determined by the singularity (pole) in the complex $(1/\nu)$-plane nearest to the origin. Beyond the disk of convergence the eddy viscosity must be continued analytically. For meromorphic functions, a very robust method of analytic continuation is by Padé approximants (Baker 1975).

4.1.1. *Practical implementation by Padé approximants*

In this subsection we specialize to basic flow with three- or six-fold symmetry such that the eddy viscosity is isotropic. We then obtain, instead of (30)

$$\nu_E = \nu+\sum_{n=1}^{\infty}\nu_E^{(n)}\nu^{-n}. \tag{31}$$

We observe that, by use of (22) and of the recursive relation (29), it is easy to give exact expressions for the first few Taylor coefficients $\nu_E^{(n)}$. For $n = 1$ and 2 they will be found in Appendix C. Observe that the first correction to the molecular viscosity (already given in DF) is always positive and has been used in particular to check subsequent numerical calculations. As shown by Sivashinsky & Frenkel (1992) this correction can be negative for time-dependent flows.

Obtaining accurate numerical values for a large number of Taylor coefficients is particularly simple when *the stream-function of the basic flow is a trigonometric* *polynomial* in the variables x_1 and x_2, i.e. has a finite number of Fourier harmonics. It then follows from the recursion relation (29) that all the f_n are trigonometric polynomials with increasingly many harmonics. Specifically, let K denote the maximum wavenumber (modulus of wave vector) present in the basic stream function Ψ. Then, the largest wavenumber present in the calculation of the nth Taylor coefficient of the eddy viscosity is less or equal to $(n+1)K$. We may then calculate the f_n by finitely truncated Fourier series, provided that the maximum wavenumber is larger than $(n+1)K$. In Fourier space, differentiation and ∂^{-2} operators are multiplications, while products with Ψ or $\partial^2\Psi$ are finite convolution sums. Such a calculation is thus free of truncation errors and has only round-off errors (so, it is better to perform it with 64-bit arithmetic).

For simple choices of the basic stream function Ψ, such as discussed in §5, up to 50 terms in the series for the eddy viscosity are easily obtained, but for the purpose of identifying negative eddy viscosities only about half as many are needed. The calculation can be further simplified by using the fact that only terms with odd powers of ν^{-1} are present when the basic flow has symmetrical streamlines (this is proved in Appendix E).

4.2. *Spectral method*

We now return to the general case of *time-dependent* basic flow Ψ. The unique space–time-periodic and zero-mean-value solutions of (23) and (24) can be obtained by numerically solving these time-dependent partial differential equations, starting from initial data of zero mean value (e.g. from zero) and letting the solution relax to convergence. Given the spatial periodicity, we can solve the equations by a (pseudo-) spectral method (Gottlieb & Orszag 1977). The details of the spectral method are standard. We use a slaved-frog temporal scheme (Frisch, She & Thual 1986*b*) with alias-removal. Since the Reynolds number for which we shall obtain negative eddy viscosities are not particularly large (typically less than 10), the method is not very demanding as far as resolution is concerned. For the cases to be reported in subsequent Sections, we have worked with resolutions from 64^2 to 512^2 and found the former to be always adequate: errors on the eddy viscosity are then about 10^{-4} in absolute terms. At the higher resolutions the energy spectra (in Fourier space) always have very conspicuous exponentially decaying tails. The actual implementations were running on a CM-200. Convergence is achieved in a few hundred to a few thousand time steps, depending on the basic flow. For 64^2 resolution such calculations are well within the capability of ordinary workstations. The same numerical method is used to check for the possibility of small-scale instabilities. This is done by choosing random initial conditions and looking for possible growing modes. A somewhat less reliable method, but which requires no additional calculations, for detecting small-scale instabilities is to start with zero initial conditions and hope that the right-hand sides of (23) and (24) will excite whatever unstable modes there may be. For the case of the 'hexagonal decorated flow' described in §5, we have systematically used the former, more reliable, method in order to make sure that negative eddy viscosity is present without any small-scale instability.

We finally make some comments on the case of basic flow with three- and six-fold rotational symmetry (which ensure isotropy of the eddy viscosity). The spectral method can be implemented for arbitrary values of the period L_1 in x_1 and the period L_2 in x_2. For three- and six-fold symmetry, the Fourier modes should be on a regular triangular lattice. This can be done by taking, e.g. $L_1 = 2\pi$ and $L_2 = 2\pi/\surd 3$. The Fourier modes are then of the form

$$\boldsymbol{k} = (2\alpha+\beta)\,\boldsymbol{e}_1+\beta\surd 3\boldsymbol{e}_2, \tag{32}$$

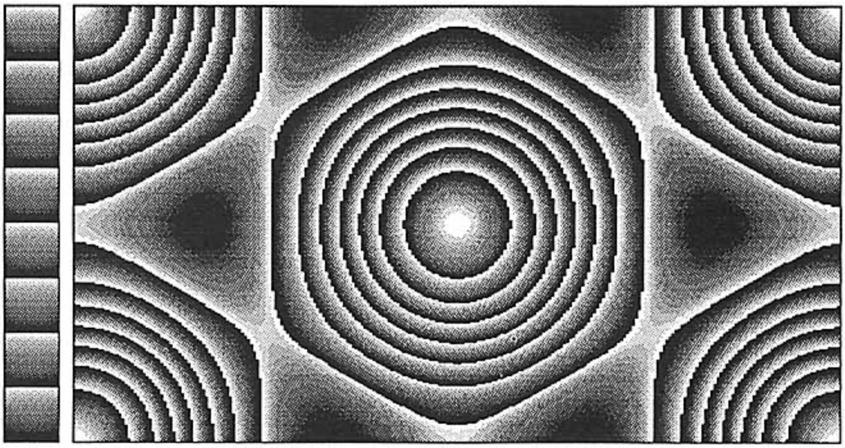

FIGURE 1. Streamlines (grey-coded) of the 'single-wavenumber flow', given by (33), the eddy viscosity of which is always positive.

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the unit vectors in the x_1 and x_2 directions, and where α and β are arbitrary-signed integers. To ensure six-fold rotational symmetry, Fourier modes of the basic flow Ψ which can be obtained by rotations of $\pi/3$ have the same Fourier amplitudes. The amplitude should be real, because the amplitude at the opposite wave vectors, which can be obtained by three rotations of $\pi/3$, should also be its complex conjugate. Thus, only the modes located in the sector of angle $\pi/3$ may be chosen independently. For three-fold symmetry, the angular sector is again of width $\pi/3$, but the amplitudes are complex numbers.

5. Simple flows with isotropic eddy viscosity

In this Section we consider only time-independent basic flow with six-fold rotational symmetry. We shall then calculate the (isotropic) eddy viscosity, using the analytical continuation method of §4.1 and the spectral method of §4.2. These methods are completely independent; this will thus provide a useful check on our results.

5.1. *Single-wavenumber flows*

The simplest flow with the required parity-invariance and six-fold symmetry has a single wavenumber and is unique up to a translation, a rotation and rescaling. It may be obtained by setting $\alpha = 1$ and $\beta = 0$ in (32). In the physical space it reads

$$\Psi(x_1, x_2) = \cos 2x_1 + \cos(x_1 + \sqrt{3}\, x_2) + \cos(x_1 - \sqrt{3}\, x_2). \tag{33}$$

This flow is shown in figure 1. Since the flow has symmetrical streamlines, the series expansion of the eddy viscosity (31) contains only odd powers of $1/\nu$ (see Appendix E). The Taylor coefficients of ν^{-n} for n up to 19 are given in table 1. It is seen that the first two non-vanishing coefficients $n = 1$ and $n = 3$ are positive and then the coefficients alternate in sign (so that the nearest singularity is on the imaginary axis). The eddy viscosity obtained by the [10/10] diagonal Padé approximant (actually [5/5] in the variable $1/\nu^2$) is shown in figure 2.

For the same flow, we have calculated the eddy viscosity by the spectral method of §4.2 for various values of ν. The results are shown on the same figure 2 and agree fully

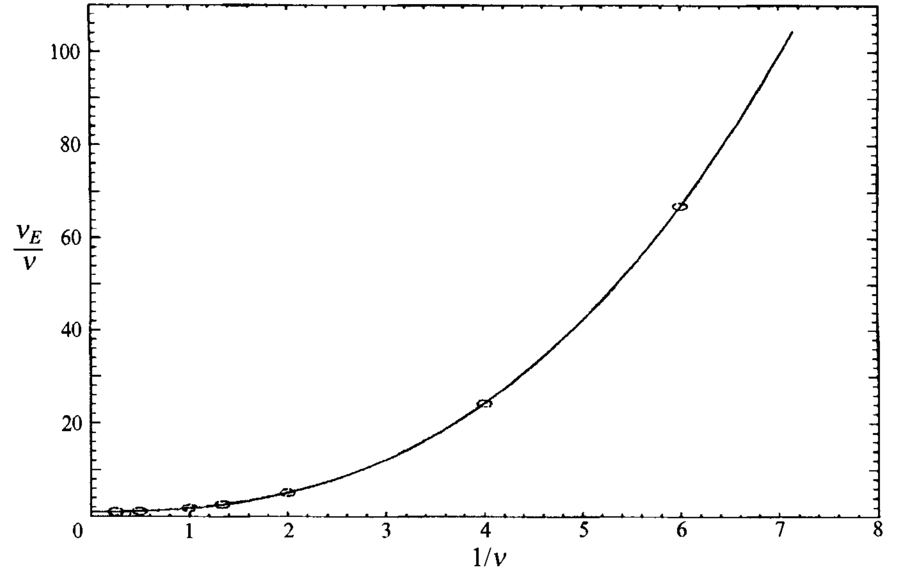

FIGURE 2. Eddy viscosity for the 'single-wavenumber flow'. ν_E/ν is plotted against $1/\nu$. The continuum line is from the Padé approximant and the circles from the spectral calculations.

n	$\nu_E^{(n)}$
1	$7.500000000 \times 10^{-1}$
3	$9.375000000 \times 10^{-2}$
5	$-9.745695153 \times 10^{-3}$
7	$1.521227229 \times 10^{-3}$
9	$-2.758184968 \times 10^{-4}$
11	$5.430174455 \times 10^{-5}$
13	$-1.125837306 \times 10^{-5}$
15	$2.412329984 \times 10^{-6}$
17	$-5.276340282 \times 10^{-7}$
19	$1.168710794 \times 10^{-7}$
21	$-2.608521638 \times 10^{-8}$

TABLE 1. Taylor coefficients $\nu_E^{(n)}$ for the single-wavenumber flow

with the Padé results. Clearly, the 'single-wavenumber flow' is not a good candidate for obtaining negative eddy viscosities.

We have also tried single-wavenumber flows with three- rather six-fold rotational symmetry, which are linear combinations of (33) and of a similar expression with each cos replaced by a sin. No case of negative eddy viscosity was found.

5.2. *The decorated hexagonal flow*

The failure to obtain negative eddy viscosities with single-wavenumber flows may have something to do with the first two non-vanishing corrections to the molecular viscosity (in powers of the Reynolds number) being positive. The first correction is always positive for time-independent flow, but there is no *a priori* constraint on the subsequent ones. We have therefore investigated flows having a few different wavenumbers, trying to get the second and the third correction to be negative. This had led us to the flow

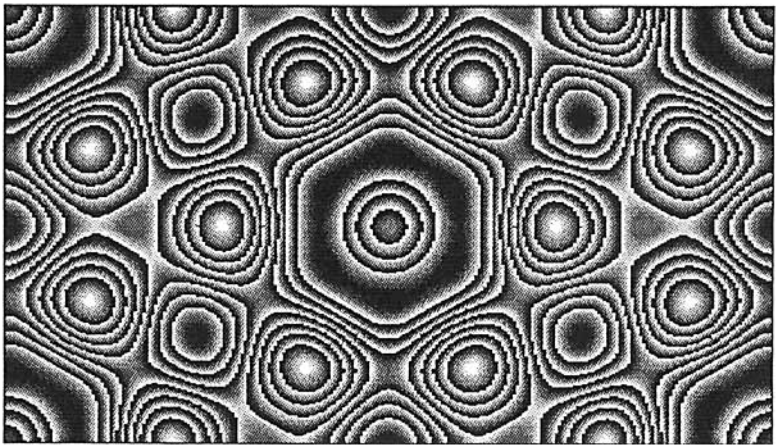

FIGURE 3. Streamlines of the 'decorated hexagonal flow', given by (34), which has a negative eddy viscosity.

n	$\nu_E^{(n)}$
1	$7.500000000 \times 10^{-1}$
3	$-3.419459887 \times 10^{-1}$
5	$-1.361038366 \times 10^{-1}$
7	$9.946897685 \times 10^{-2}$
9	$-2.023193057 \times 10^{-2}$
11	$-8.393763844 \times 10^{-3}$
13	$8.520740997 \times 10^{-3}$
15	$-5.293578026 \times 10^{-3}$
17	$4.255837684 \times 10^{-3}$
19	$-4.028466334 \times 10^{-3}$
21	$3.598818877 \times 10^{-3}$

TABLE 2. Taylor coefficients $\nu_E^{(n)}$ for the decorated hexagonal flow

$$\begin{aligned}\Psi(x_1, x_2) = & -\tfrac{1}{2}[\cos 2x_1 + \cos(x_1 + \sqrt{3}\,x_2) + \cos(x_1 - \sqrt{3}\,x_2)] \\ & + \tfrac{1}{2}[\cos(4x_1 + 2\sqrt{3}\,x_2) + \cos(5x_1 - \sqrt{3}\,x_2) + \cos(x_1 - 3\sqrt{3}\,x_2)] \\ & - \tfrac{1}{2}[\cos(4x_1) + \cos(2x_1 + 2\sqrt{3}\,x_2) + \cos(2x_1 - 2\sqrt{3}\,x_2)] \\ & + \tfrac{1}{2}[\cos(4x_1 - 2\sqrt{3}\,x_2) + \cos(5x_1 + \sqrt{3}\,x_2) + \cos(x_1 + 3\sqrt{3}\,x_2)], \end{aligned} \tag{34}$$

the streamlines of which are shown in figure 3. This flow, which has three distinct wavenumbers, will be called 'decorated hexagonal flow'. It again has symmetrical streamlines and thus only odd powers of $1/\nu$ in the expansion of the eddy viscosity. In table 2 we show the Taylor coefficients for the eddy viscosity up to ν^{-21}. The second and third non-vanishing correction to the molecular viscosity is seen to be negative. Because this flow achieves very low and actually negative eddy viscosities it is better to show the results for the Padé approximant ([10/10] diagonal) and the spectral calculation in tabular form (table 3).

The agreement between the Padé approximant and the spectral calculations at both moderate and high resolution is again remarkable and leaves little doubt that *the eddy viscosity changes sign* when the molecular viscosity drops below $\nu_c \approx 0.58$. Note that the radius of convergence of the series (31) in powers of $1/\nu$ is ≈ 1.1, so that the value

ν	Spectral 64^2 ν_E	Spectral 256^2 ν_E	Padé prediction ν_E
1	1.348	1.348	1.348
0.8	0.961	0.961	0.961
0.7	0.642	0.642	0.642
0.6	0.134	0.134	0.134
0.59	0.066	0.066	0.966
0.58	−0.007	−0.007	−0.007
0.57	−0.085	−0.085	−0.085
0.56	−0.169	−0.169	−0.169
0.55	−0.259	−0.259	−0.260

TABLE 3. Eddy-viscosity ν_E for the decorated hexagonal flow

for which the eddy viscosity changes sign is well beyond the radius of convergence. We have also carefully checked that no small-scale instability develops as long as $\nu > 0.549$.

6. How common is negative eddy viscosity?

Having observed one particular flow with negative eddy viscosity, it is natural to ask if this property is exceptional in the class of parity-invariant flow with six-fold rotational symmetry. To try and answer this, we have analysed a large number of different time-independent flows and in each case varied the molecular viscosity, starting from $\nu = 2$ and halving it until one of the following happens: (i) the eddy viscosity becomes negative, (ii) the eddy viscosity begins to increase, (iii) a small-scale instability appears. In the third event, a refined search in between the last two values is made. The random basic flows were all Gaussian and thus statistically characterized by their energy spectra. The latter were chosen to be of the form k^{-n} with a cutoff at $K = 7$, beyond which the energy spectrum is zero. This search, carried out on about 500 flows, revealed that around 30 % of the flows eventually developed a negative eddy viscosity, when lowering the molecular viscosity. Specifically, for n between 1 and 3 the percentage stayed around 30 (within the statistical scatter inherent to such a procedure). For $n = 10$, which is a very steep spectrum, out of one hundred trials, not a single case of negative eddy viscosity was obtained. This is not surprising, since such flows have most of their excitation in the lowest-wavenumber modes, and we know that the single-wavenumber flow has a positive eddy viscosity. In the course of our investigation, we have also found some rather amusing flows with negative eddy viscosities, such as the one shown in figure 4.

We have tried to find a simple rule for guessing if a flow can have negative eddy viscosity. Visual inspection of a number of such flows has revealed that they usually possess regions of rather closely packed streamlines, such as the roughly circular structure encircling the 'gear' in figure 4. So, at least locally, such flows resemble the Kolmogorov flow which is known to have a negative (albeit highly anisotropic) eddy viscosity.

Anyway, we conclude that the phenomenon of two-dimensional isotropic negative eddy viscosity is quite common.

We observe that the existence of isotropic negative eddy viscosity is still an open problem in three dimensions. The three-dimensional case is also much richer; for

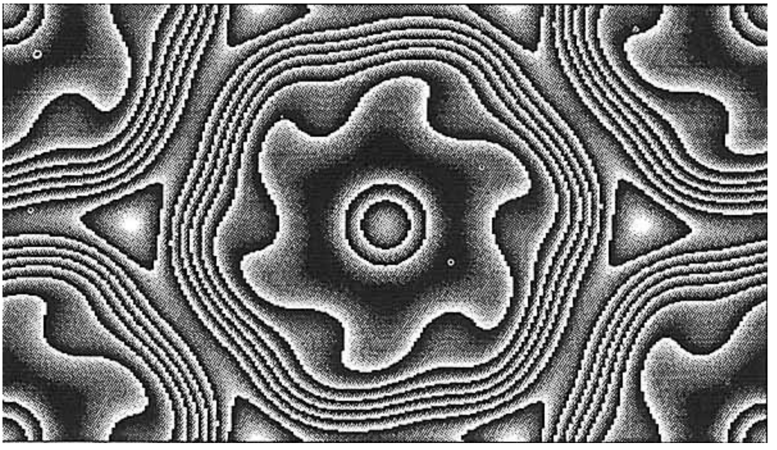

FIGURE 4. One of the many Gaussian flows with negative eddy viscosity which has six-fold symmetry and a k^{-1} energy spectrum.

example the eddy viscosity can become complex, in the sense that oscillations are permitted (Wirth 1993).

7. Time-dependent flow

The spectral method of §4.2 applies equally well to time-dependent basic flow. Sivashinsky & Frenkel (1992) have observed that for time-dependent flow the first correction to the molecular viscosity in powers of the Reynolds number may be negative. This is the case, in particular, for the flow considered in their paper, which is

$$\Psi = \frac{2\surd 3}{3}\cos\omega t\left\{\sin x_1 - 2\sin\tfrac{1}{2}x_1 \sin\frac{\surd 3}{2}x_2\right\}. \tag{35}$$

Their flow, shown in figure 5, has three-fold rotational symmetry and thus an isotropic eddy viscosity. We have calculated the latter by the spectral method for various values of the Reynolds number $R = 1/\nu$ and of the modulation frequency ω. The sign of the results in the (R, ω)-plane are shown in figure 6. Sivashinsky & Frenkel did not attempt to calculate the eddy viscosity exactly. Instead, they expanded it in powers of $R = 1/\nu$ for small R and assumed $\tilde{\omega} = \omega R$ to be finite. This gave (in our notation)

$$\nu_E = \nu + \nu^{-1}\frac{1}{1+\tilde{\omega}^2}\left(\frac{1}{1+\tilde{\omega}^2} - \frac{1}{2}\right) + O(\nu^{-2}). \tag{36}$$

Using only the first two terms they obtained in the (R, ω)-plane a separatrix between negative and positive eddy viscosities. This qualitative feature is common to their low-order expansion and to our accurate calculation. On the other hand, there is exact quantitative agreement only for small Reynolds numbers when the eddy viscosity is dominated by its molecular contribution and thus cannot be negative. At higher Reynolds numbers there is some discrepancy, but it does not exceed 10%, so that the low-order Reynolds-number approximation (36) appears to be rather good.

We saw in §6 that negative eddy viscosity is a very frequent phenomenon with time-independent flow. Thus, time-dependence is clearly not needed.

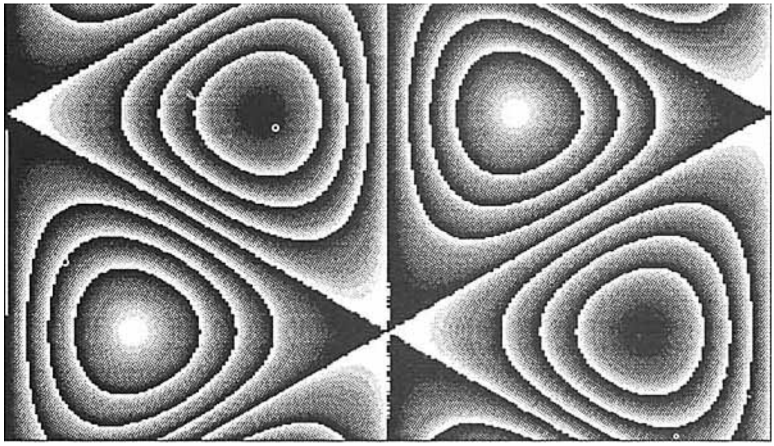

FIGURE 5. A snapshot of the Sivashinsky & Frenkel (1992) flow.

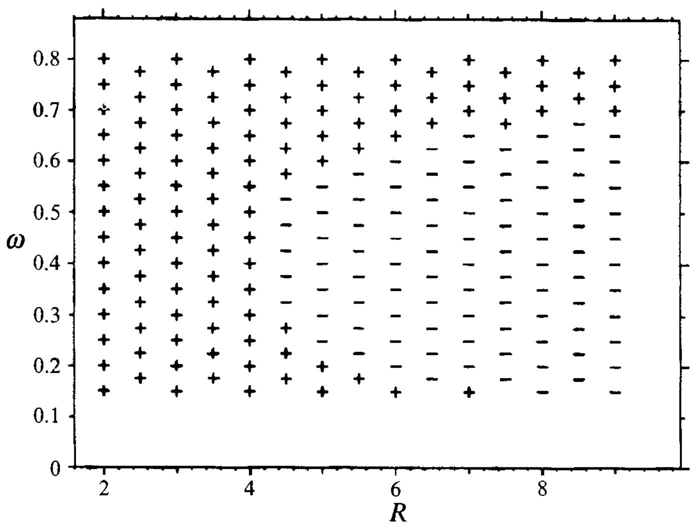

FIGURE 6. Sign of the eddy viscosity for the Sivashinsky & Frenkel (1992) flow. R is the Reynolds number and ω the modulation frequency.

7.1. *Flows δ-correlated in time*

There is an instance, other than low Reynolds numbers, where the eddy viscosity can be calculated in explicit analytic form, namely for Gaussian basic flow which is δ-correlated in time (white noise). To define precisely what we mean, we begin with a basic flow Ψ (not yet δ-correlated) which is a Gaussian random function of space and time with zero mean and correlation function

$$\langle \Psi(x,t)\,\Psi(x',t')\rangle = \Gamma_{\Psi}(x-x',t-t'). \tag{37}$$

The correlation function Γ_{Ψ} is assumed to depend only on $x-x'$ (homogeneity) and on $t-t'$ (stationarity), to be even in $t-t'$ (time-reversal symmetry) and to decrease sufficiently fast at large space or time separations (mixing). Let η be a small expansion parameter (chosen independently of the ϵ in the multiscale expansion). We define

$$\Psi_{\eta}(x,t) = \frac{1}{\eta}\,\Psi\!\left(x,\frac{t}{\eta^2}\right). \tag{38}$$

It is easily seen that, when $\eta \to 0$, the function Ψ_η becomes δ-correlated in time:

$$\lim_{\eta \to 0} \langle \Psi_\eta(\boldsymbol{x}, t)\, \Psi_\eta(\boldsymbol{x}', t') \rangle = \delta(t-t')\, \Gamma_S(\boldsymbol{x}-\boldsymbol{x}'), \tag{39}$$

$$\Gamma_S(\boldsymbol{x}-\boldsymbol{x}') \equiv \int_{-\infty}^{+\infty} \Gamma(\boldsymbol{x}-\boldsymbol{x}', \tau)\, \mathrm{d}\tau. \tag{40}$$

In other words, δ-correlated flows are here obtained as limits of flows with a very short correlation time $O(\eta^2)$. This formulation avoids mathematical problems (which must otherwise be overcome by using Ito calculus). It also provides us with an additional expansion parameter η to simplify the solution of the auxiliary problems.

As we have already stated, the multiscale formalism is essentially the same for periodic and random functions, except that the angular brackets $\langle \cdot \rangle$ now denote ensemble averages. In particular (22)–(25) remain true. It is shown in Appendix D that the auxiliary problems can now be solved exactly in the limit $\eta \to 0$, thanks to an additional multiscale expansion in the time variable. Other techniques for solving linear equations with white-noise coefficients could be used as well (see e.g. Brissaud & Frisch 1974).

The final result for the eddy viscosity tensor is given by (D 18) is Appendix D. When the basic flow is isotropic, $\nu_E = \nu$. Thus, isotropic two-dimensional incompressible random flow, with white-noise time-dependence, has an eddy viscosity equal to its molecular viscosity.

Several remarks are now in order. First, we note that AKA-terms (first-order tensors in large-scale derivatives) vanish for δ-correlated flow, irrespective of parity-invariance. Second, we observe that the vanishing of the correction to the molecular viscosity is a special property of two dimensions. In higher dimensions, otherwise under the same conditions, the correction to ν is strictly positive. Third, we observe that the vanishing of the correction is already implicitly contained in Kraichnan (1976). In that paper, he uses a closure, known as Test Field Model (TFM), involving a relaxation time θ_{kpq} for triple correlations. It is easily checked that in the δ-correlated case, the TMF closure is exact with a θ_{kpq} that is a constant. The correction to the molecular viscosity (Kraichnan 1976, equation (4.6)) is then found to vanish. Fourth, we observe that the eddy viscosity plays an important role in renormalization group calculations (Forster, Nelson & Stephen 1977; Fournier & Frisch 1983), which produce particularly non-trivial results in two dimensions. In the renormalization formalism, the external driving force is δ-correlated, but the resulting flow is not, so that the correction to the molecular viscosity does not vanish.

8. Nonlinear dynamics of mirror-symmetric basic flow

In subsequent Sections we address the question of the large-scale dynamics, when the amplitude of the large-scale flow is not sufficiently small to permit linearization. This may happen either because a negative eddy-viscosity instability is amplifying the large-scale flow up to the point where nonlinearities are relevant (§8.2) or because the initial amplitude was taken sufficiently large (§8.1). We shall see that there is a basic difference between the cases of chiral and non-chiral forcing. In the present Section, we begin with the latter. We shall also assume henceforth that the flow has at least six-fold rotational symmetry.

8.1. *The case of positive eddy viscosity*

We assume here mirror-symmetry forcing (more precisely, a mirror-symmetric basic flow), six-fold rotational symmetry and an eddy viscosity that is positive. Consider the Navier–Stokes equation (6) for the perturbation ψ to the basic flow Ψ. We now ask: when does the nonlinear term $J(\partial^2\psi, \psi)$ become comparable with the eddy-viscosity term $\nu_E \partial^2 \partial^2 \psi$? Since the large-scale motion has spatial scale $O(\epsilon^{-1})$, all spatial derivatives contribute a factor $O(\epsilon)$. Denoting by $[\psi]$ a typical amplitude of the perturbation ψ, we find that the Jacobian term is $O(\epsilon^4[\psi]^2)$ and the eddy-viscosity term $O(\epsilon^4[\psi])$. These terms are comparable when $[\psi] = O(\epsilon^0)$. In Appendix B (where the mirror-symmetry and the chiral cases are handled together), it is shown that the multiscale expansion can be modified to incorporate nonlinear terms. This is actually rather straightforward because, fortunately, the auxiliary problems to be solved remain the same linear equations as before. The equation arising from solvability at order ϵ^4 in the mirror-symmetric case is

$$\partial_T \nabla^2 \psi^{(0)} + a\mathcal{J}(\nabla^2 \psi^{(0)}, \psi^{(0)}) = \nu_E \nabla^2 \nabla^2 \psi^{(0)}, \tag{41}$$

where $\mathcal{J}$ denotes the Jacobian in slow (large-scale) variables. The eddy viscosity ν_E has the same expression (22) as in the linear case. The coefficient a is given in Appendix B.

It may be surprising that a is, in general, not equal to one, since the latter is needed to ensure Galilean invariance. As noted by Yakhot, Bayley & Orszag (1986), who attribute it to Sivashinsky (1985), Galilean invariance is lost because of the presence of an external force. Indeed, when going to a reference frame moving with the velocity $\boldsymbol{u}$, the force $\boldsymbol{f}(\boldsymbol{x}, t)$ becomes $\boldsymbol{f}(\boldsymbol{x}-\boldsymbol{u}t, t)$. Only a δ-correlated force will stay invariant. Of course, Galilean invariance can be easily restored by absorbing the coefficient a into a rescaled time variable, as is done in the theory of lattice gases (Frisch *et al.* 1986*a*, 1987*a*). Actually, it was pointed out by Yakhot *et al.* (1986) that there is considerable analogy between hydrodynamics with small-scale forcing and lattice gases, as far as the form of the large-scale equations is concerned. It may be shown that the coefficient a, which is the analogue of the Galilean factor in lattice gases, vanishes under certain conditions (Gama & Vergassola 1993). Note that the vanishing of the advective nonlinearity for the case of the Kolmogorov flow is due to the fact that the large-scale flow depends on a single coordinate. Observe that, as long as $a \neq 0$, it does not matter much what its sign may be, since the latter can be changed by just redefining the sign of the velocity. We conclude that, in general, the large-scale dynamics of flow with positive eddy viscosity is governed by the ordinary two-dimensional decaying Navier–Stokes equation, which is known to have regular solutions for all times.

8.2. *The case of negative eddy viscosity*

The case of negative eddy viscosity has already been investigated in detail for the Kolmogorov flow (Nepomnyashchy 1976), who showed that when the eddy viscosity is marginally negative (i.e. $O(\epsilon^2)$), the large-scale dynamics is governed by a one-dimensional Cahn–Hilliard equation. In our notation, this equation reads

$$\partial_T \psi^{(0)} = -\nabla_1 \{(1-|\nabla_1 \psi^{(0)}|^2)\nabla_1 \psi^{(0)}\} - \nabla_1^4 \psi^{(0)}. \tag{42}$$

There is a linearly unstable wavenumber band at large scales and the nonlinearity saturates the instability since it tends to bring the eddy viscosity back to positive values. In order to obtain the Cahn–Hilliard equation, the approximate scaling (assume spatial scales $O(1/\epsilon)$) is ϵ^{-2} for the time variable and ϵ^{-1} for the large-scale stream function.

When the eddy viscosity is marginally negative, but *isotropic*, the above picture is somewhat modified because, in general, the advective nonlinearity $\mathscr{J}(\nabla^2\psi^{(0)}, \psi^{(0)})$ does not vanish. Hence, the scaling is modified. When the eddy viscosity is negative and $O(\epsilon^2)$, timescales are $O(\epsilon^{-4})$ and the stream function is $O(\epsilon^2)$, as is revealed by balancing nonlinear and negative eddy-viscosity terms. Just as for the Kolmogorov flow, linear terms involving two additional space derivatives must be included. Such terms, involve six spatial derivatives acting on the large-scale stream function. This raises two difficulties. First, the isotropy of *sixth-order tensors* is not guaranteed by six-fold rotational symmetry. The second difficulty is that the term with six spatial derivatives need not be dissipative. If it is not, then perturbation theory is not applicable (there is a 'first-order' transition). Fortunately, we found numerically that for the decorated hexagonal flow the six-order operator is dissipative and very close to isotropy (the discrepancies are about 1 %).

Exact isotropy can be ensured if the basic flow has five-fold rotational symmetry. This is however not compatible with periodicity: such flows are, at best, quasi-periodic. Another way is to work with random homogeneous and isotropic flows. Quasi-periodic and random flows present new difficulties. For example, the compactness of the operator B defined in (27) may be lost. Such questions are beyond the scope of the present paper.

As long as isotropy holds, the equation for the large-scale flow, which arises from solvability at order ϵ^8, reads (see derivation in Appendix B)

$$\partial_T \nabla^2\psi^{(0)} + a\mathscr{J}(\nabla^2\psi^{(0)}, \psi^{(0)}) = -\mu_4 \nabla^2\nabla^2\psi^{(0)} - \mu_6 \nabla^2\nabla^2\nabla^2\psi^{(0)}. \tag{43}$$

The coefficient a is again due to the lack of Galilean invariance, as discussed in §8.1. The coefficient μ_4 requires the evaluation of $(\partial\nu_E/\partial\nu)_{\nu=\nu_c}$, where ν_c is the 'critical' value for the viscosity at which $\nu_E = 0$.

Equation (43), when μ_4 and μ_6 are positive and $a \neq 0$, is equivalent to the Navier–Stokes–Kuramoto–Sivashinsky (NSKS) equation.

The NSKS equation has been studied numerically on a Connection Machine CM-2 by Gama *et al.* (1991). Their findings are briefly summarized here. Up to millions of time steps at the resolution 256^2 and tens of thousands at the resolution 1024^2 were performed. A linear growth phase, a disorganized inverse cascade phase and a structured vortical phase were successively observed. In the vortical phase, monopolar and multipolar structures were found to proliferate and displayed strongly depleted nonlinearities. Similar observations have been made for the forced two-dimensional Navier–Stokes equation (Legras, Santangelo & Benzi 1988). Gama *et al.* (1991) also found that after extremely long times (thousands of eddy-turnover times), a filamentation phenomenon starts which leads to the abrupt appearance of very fine scales. This may be a signal that the NSKS equation is an approximation to the large-scale dynamics of the original problem which is not uniform in time. So, eventually, higher-order nonlinear terms, may have to be incorporated.

9. Nonlinear dynamics of chiral flow

Let us now assume that the force and the basic flow Ψ are chiral (not mirror-symmetric), while still possessing six-fold rotational symmetry. The main consequence (proven in Appendix B) is that a new nonlinear term appears. For example, when the eddy viscosity is positive, instead of (41), we obtain

$$\partial_T \nabla^2\psi^{(0)} + a\varepsilon_{ij}(\nabla_i \nabla^2\psi^{(0)}) \nabla_j \psi^{(0)} + c\nabla_i [(\nabla^2\psi^{(0)}) \nabla_i \psi^{(0)}] = \nu_E \nabla^2\nabla^2\psi^{(0)}. \tag{44}$$

The second term on the left-hand side is the usual Jacobian term, written here with the antisymmetric ε_{ij} tensor in order to bring out that the other nonlinear term does not involve such a tensor. The third term is not compatible with mirror symmetry, which requires a change in the sign of the stream function. The same 'chiral term' must be added to (43), when the eddy viscosity is marginally negative.

A simple example of chiral flow is the decorated hexagonal flow of figure 3, which possesses symmetric streamlines, but not mirror symmetry. At the critical value $\nu = \nu_c \approx 0.58$, where the eddy viscosity vanishes, this flow has $a = 1.88$ and $c = 0.52$.

The same equation (44) applies also when the force is random homogeneous stationary isotropic and chiral. Note however that this rules out Gaussian forces. Indeed, in two dimensions, a force of zero divergence may be written in terms of a scalar function $f_i = \varepsilon_{ij}\partial_j\phi$. If ϕ is Gaussian and of zero mean, it is completely characterized by its correlation function $\langle\phi(\boldsymbol{x},t)\phi(\boldsymbol{x}',t')\rangle$. The latter, if homogeneous and isotropic, is also obviously mirror-symmetric.

Equation (44) is not a standard problem. For example, it has no energy or enstrophy conservation by nonlinear terms, neither is there any need for this, since there is an interplay of small-scale and large-scale energy and enstrophy. It is not even clear that (44) is well posed for more than a finite time. It is certainly not when the viscous term is ignored. Indeed, contrary to what happens for the two-dimensional incompressible Euler equation, the solutions of (44) typically blow up after a finite time when the viscous term is ignored. To show this, let us define the large-scale vorticity

$$\omega_{LS} = -\nabla^2\psi^{(0)}, \tag{45}$$

and two large-scale velocities

$$\boldsymbol{v}_{LS} = a(\nabla_2\psi^{(0)}, -\nabla_1\psi^{(0)}), \quad \boldsymbol{v}'_{LS} = c(\nabla_1\psi^{(0)}, \nabla_2\psi^{(0)}). \tag{46}$$

We may then write (44) with $\nu_E = 0$ as

$$\partial_T\omega_{LS} + \boldsymbol{v}_{LS}\cdot\boldsymbol{\nabla}\omega_{LS} + \boldsymbol{\nabla}\cdot(\boldsymbol{v}'_{LS}\,\omega_{LS}) = \partial_T\omega_{LS} + (\boldsymbol{v}_{LS}+\boldsymbol{v}'_{LS})\cdot\boldsymbol{\nabla}\omega_{LS} - c\omega_{LS}^2 = 0. \tag{47}$$

Defining the 'Lagrangian derivative' $\mathrm{D}_T \equiv \partial_T + (\boldsymbol{v}_{LS}+\boldsymbol{v}'_{LS})\cdot\boldsymbol{\nabla}$, we obtain

$$\mathrm{D}_T\,\omega_{LS} = c\omega_{LS}^2, \tag{48}$$

the solution of which blows up after a finite time, when ω_{LS} has initially the same sign as the coefficient c.

There are some instances where (44), including the viscous term, can be reduced to known problems. The simplest case is when the large-scale flow depends on a *single coordinate*, say X_1. The Jacobian term then vanishes and we are left with

$$\partial_T\nabla_1^2\psi^{(0)} + \tfrac{1}{2}c\nabla_1^2(\nabla_1^2\psi^{(0)})^2 = \nu_E\nabla_1^2\nabla_1^2\psi^{(0)}. \tag{49}$$

Assuming that $\psi^{(0)}$ has no uniform component, we obtain

$$\partial_T\psi^{(0)} + \tfrac{1}{2}c(\nabla_1\psi^{(0)})^2 = \nu_E\nabla_1^2\psi^{(0)}. \tag{50}$$

This is the well-known Burgers equation (see e.g. Burgers 1974) written for a velocity potential. The 'Burgers velocity' has then a single component in the X_1 direction, $u = \nabla_1\psi^{(0)}$. Thus, the 'Burgers velocity' is obtained from the hydrodynamical large-scale velocity

$$\boldsymbol{u} = (\nabla_2\psi^{(0)}, -\nabla_1\psi^{(0)}) = (0, -\nabla_1\psi^{(0)}), \tag{51}$$

by a rotation of $\pi/2$.

Let us now consider the case when the large-scale flow depends only on a radial coordinate $\psi^{(0)} = \psi^{(0)}(R, T)$ with $R = (X_1^2 + X_2^2)^{\frac{1}{2}}$, so that the Jacobian term vanishes again. Defining

$$\chi = R\frac{\partial \psi^{(0)}}{\partial R}, \quad \rho = R^2, \tag{52}$$

we obtain

$$\partial_T \chi + c\frac{\partial \chi^2}{\partial \rho} = 4\nu_E \rho \frac{\partial^2 \chi}{\partial \rho^2}. \tag{53}$$

This equation has the same nonlinearity as the Burgers equation, but a ρ-dependent viscosity. Hence, if we have a large-scale flow initially consisting of nearly circular well-separated vortex patches and then let it evolve under the action of the chiral term, the patches having a vorticity of the same sign as the coefficient c will be enhanced and concentrated, while those of opposite sign will be fattened and attenuated.

Finally, we remark that Moffatt (1983) has found a chiral effect, a contribution proportional to ε_{ij}, in the eddy-diffusivity tensor D_{ij} for a passive scalar. (He refers to chiral flow as having a preferred 'sense of turning'.) This effect does not modify the large-scale transport equation, since in the latter the eddy diffusivity is contracted with two gradients. Still, there is a contribution of the chiral term to the flux of concentration of the passive scalar and this flux could be measurable.

10. Concluding remarks

We have shown that in two dimensions negative isotropic eddy viscosity is a very common phenomenon, even in time-independent flow. Kraichnan (1976) tried to give a phenomenological interpretation of this phenomenon, the existence of which was suggested by closure calculations. He used the fact that there exist simple flows, depending on just one spatial coordinate, such as the Kolmogorov flow

$$(u_1, u_2) = (1/\nu)(\gamma \sin x_2, 0), \quad p = \text{const.}, \quad \boldsymbol{f} = (\gamma \sin x_2, 0), \quad \gamma > 0, \tag{54}$$

which have a negative but highly anisotropic eddy viscosity. Such flows can be used as local building stones of random homogeneous and isotropic 'superflows', by slowly varying in space and time the parameters of the one-dimensional flows. The problem of finding the eddy viscosity of the superflow is then at least as difficult as finding the effective diffusivity of a material with periodically or randomly varying diffusivity, a problem which in more than one dimensions has no analytical solution (Bensoussan *et al.* 1978). This may be why we have not yet found a convincing phenomenological interpretation of *isotropic* negative eddy viscosity.

Because isotropic negative eddy viscosity is such a common phenomenon, two-dimensional flow with small-scale forcing will often develop large-scale instabilities, when the Reynolds number is increased. Just above the threshold of instability, the large-scale dynamics is governed by the Navier–Stokes–Kuramoto–Sivashinsky equation (43), which is known to produce monopolar and multipolar vortical structures with strongly depleted nonlinearities (Gama *et al.* 1991).

More exotic behaviour is expected when the small-scale forcing is chiral so that the large-scale equation (44) contains a term that, in special cases, is the same as in the Burgers equation. The fate of large-scale vortices will then depend on their cyclonicity, one kind being enhanced and the other one attenuated. Observe that in the usual Navier–Stokes equation any circular vortex has a vanishing nonlinearity, irrespective of its radial vorticity distribution. The determination of the latter, in such instances where it is conjectured to be universal, becomes then a highly non-trivial question (Miller 1990; Robert & Sommeria 1991). The presence of an additional chiral nonlinearity in the large-scale dynamics may be helpful in selecting the shape of such vortices, since the chiral term does not vanish for arbitrary radial dependence. (A similar observation can be made about Cahn–Hilliard terms in the mirror symmetric-case.)

The detailed study of such questions is beyond the scope of the present paper. Here, we just observe that handedness (chirality) has well-known effects in three-dimensional hydrodynamics, where it expresses the lack of parity-invariance. Examples are the linear α-effect of magnetohydrodynamics (Steenbeck, Krause & Rädler 1966) and the linear AKA-effect (Frisch *et al.* 1987*b*). In two dimensions, handedness is related to mirror-invariance and not to parity-invariance, and so its effects are seen only in nonlinear theory.

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Appendix A. Multiscale technique (linear)

In this Appendix we derive equation (21) for the dynamics of large-scale perturbations sufficiently weak so that we can use the linearized Navier–Stokes equation (rewritten here for convenience):

$$\mathscr{A}\psi \equiv \partial_t \partial^2\psi + J(\partial^2\psi, \Psi) + J(\partial^2\Psi, \psi) - \nu\partial^2\partial^2\psi = 0. \tag{A 1}$$

The solution of (A 1) is now assumed to depend on the fast variables $\boldsymbol{x}$ and t and on the slow variables $\boldsymbol{X} = \epsilon\boldsymbol{x}$ and $T = \epsilon^2 t$. Derivatives in (A 1) must then be decomposed according to

$$\partial_i \rightarrow \partial_i + \epsilon\nabla_i, \quad \partial_t \rightarrow \partial_t + \epsilon^2\partial_T. \tag{A 2}$$

The solution ψ is expanded in powers of ϵ:

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + \dots. \tag{A 3}$$

The order in ϵ of the leading term is irrelevant (because the equation is linear); thus $\psi^{(0)}$ will be arbitrarily chosen to be $O(\epsilon^0)$.

The basic flow Ψ is assumed to depend only on fast variables, to be periodic in x_1, x_2 and t and to have a centre of symmetry. Three- or six-fold rotational invariance will be assumed only at a later stage.

We now derive the first three 'auxiliary' equations (15)–(17) corresponding to orders ϵ^0, ϵ^1 and ϵ^2. Using (A 2) and (A 3) in (A 1) and identifying terms $O(\epsilon^0)$, we obtain

$$\mathrm{A}\psi^{(0)} = 0. \tag{A 4}$$

Here, A is the linearized Navier–Stokes operator $\mathscr{A}$ restricted to functions that have the same space–time periodicity as the basic flow Ψ. Equation (A 4) implies that $\psi^{(0)}$ is in the null-space of A. The latter contains only constants. Thus, $\psi^{(0)}$ depends only on slow variables. This fact will considerably simplify all the subsequent algebra, since all terms in which $\psi^{(0)}$ is differentiated with respect to fast variables will disappear.

Since we need only terms up to second order in ϵ to derive (15)–(17), we expand the various operators in $\mathscr{A}$ up to $O(\epsilon^2)$, using (A 2) and (A 3). We obtain

$$\partial_t \partial^2 \psi \rightsquigarrow \epsilon \partial_t \partial^2 \psi^{(1)} + \epsilon^2 [2\partial_t \partial \cdot \nabla \psi^{(1)} + \partial_t \partial^2 \psi^{(2)}] + O(\epsilon^3), \tag{A 5}$$

$$J(\partial^2 \psi, \Psi) \rightsquigarrow \epsilon J(\partial^2 \psi^{(1)}, \Psi) + \epsilon^2 \{J(\partial^2 \psi^{(2)}, \Psi) + \varepsilon_{\alpha\beta}(\partial_\beta \Psi)[\nabla_\alpha \partial^2 \psi^{(1)} + 2\partial_\alpha \partial \cdot \nabla \psi^{(1)}]\} + O(\epsilon^3), \tag{A 6}$$

$$J(\partial^2 \Psi, \psi) \rightsquigarrow \epsilon [J(\partial^2 \Psi, \psi^{(1)}) + \varepsilon_{i\alpha}(\partial_i \partial^2 \Psi)(\nabla_\alpha \psi^{(0)})] + \epsilon^2 [J(\partial^2 \Psi, \psi^{(2)}) + \varepsilon_{\alpha\beta}(\partial_\alpha \partial^2 \Psi)(\nabla_\beta \psi^{(1)})] + O(\epsilon^3), \tag{A 7}$$

$$\nu \partial^2 \partial^2 \psi \rightsquigarrow \nu [\epsilon \partial^2 \partial^2 \psi^{(1)} + \epsilon^2 (\partial^2 \partial^2 \psi^{(2)} + 4\partial \cdot \nabla \partial^2 \psi^{(1)})] + O(\epsilon^3). \tag{A 8}$$

Collecting in (A 5)–(A 8) terms order $O(\epsilon^1)$ and $O(\epsilon^2)$, respectively, gives

$$\mathrm{A}\psi^{(1)} = \varepsilon_{\alpha i}(\partial_i \partial^2 \Psi)(\nabla_\alpha \psi^{(0)}), \tag{A 9}$$

$$\mathrm{A}\psi^{(2)} = [-2\partial_t \partial_\alpha - \varepsilon_{i\alpha}(\partial_i \partial^2 \Psi) + 2\varepsilon_{ij}(\partial_i \Psi)\partial_j \partial_\alpha + \varepsilon_{i\alpha}(\partial_i \Psi)\partial^2 + 4\nu \partial_\alpha \partial^2] \nabla_\alpha \psi^{(1)}. \tag{A 10}$$

Observe that the solvability conditions are trivially satisfied for (A 9) and (A 10) because of the presence of fast derivatives everywhere on the left. Thanks to the independence of $\psi^{(0)}$ on fast variables, the solution of (A 9) may be written as

$$\psi^{(1)} = \boldsymbol{Q} \cdot \nabla \psi^{(0)} + \langle \psi^{(1)} \rangle, \tag{A 11}$$

where $\langle \cdot \rangle$ denotes space–time averages on fast variables, and $\boldsymbol{Q} = (Q_\alpha)$ depends only on fast variables and is taken to be the only zero-mean-value solution of

$$\tilde{\mathrm{A}} Q_\alpha = \varepsilon_{\alpha i}(\partial_i \partial^2 \Psi). \tag{A 12}$$

Similarly, and using (A 11), the solution of (A 10) reads

$$\psi^{(2)} = S_{\alpha\beta} \nabla_\alpha \nabla_\beta \psi^{(0)} + \boldsymbol{Q} \cdot \nabla \langle \psi^{(1)} \rangle + \langle \psi^{(2)} \rangle, \tag{A 13}$$

where $S_{\alpha\beta}$ is the zero-mean-value solution of

$$\tilde{\mathrm{A}} S_{\alpha\beta} = -2\partial_t(\partial_\alpha Q_\beta) - \varepsilon_{i\alpha}(\partial_i \partial^2 \Psi) Q_\beta + 2\varepsilon_{ij}(\partial_i \Psi)(\partial_j \partial_\alpha Q_\beta) + \varepsilon_{i\alpha}(\partial_i \Psi)(\partial^2 Q_\beta) + 4\nu(\partial_\alpha \partial^2 Q_\beta). \tag{A 14}$$

We now come back to the solvability conditions. There is a quick way to obtain the latter by just decomposing derivatives in (A 1), using (A 2), and taking averages, without yet expanding ψ. For this, we note that by (A 2)

$$\langle \partial_t \partial^2 \psi \rangle \rightsquigarrow \epsilon^4 \partial_T \nabla^4 \langle \psi \rangle, \tag{A 15}$$

$$\langle J(\partial^2 \psi, \Psi) + J(\partial^2 \Psi, \psi) \rangle \rightsquigarrow \epsilon^2 \varepsilon_{\alpha\beta} \langle [\nabla_\alpha (2\partial \cdot \nabla + \epsilon \nabla^2)\psi](\partial_\beta \Psi) \rangle, \tag{A 16}$$

$$\langle \nu \partial^2 \partial^2 \psi \rangle \rightsquigarrow \nu \epsilon^4 \nabla^2 \nabla^2 \langle \psi \rangle. \tag{A 17}$$

To obtain (A 16) we used

$$\langle J(f, g) \rangle = 0, \quad \langle \varepsilon_{\alpha\beta}(\partial_\alpha \partial^2 \Psi)(\nabla_\beta \psi) + \varepsilon_{\alpha\beta}(\nabla_\alpha \partial^2 \psi)(\partial_\beta \Psi) \rangle = 0. \tag{A 18}$$

Using (A 15)–(A 17) in (A 1) and averaging, we obtain

$$\epsilon^4 \partial_T \nabla^2 \langle \psi \rangle + \epsilon^2 \varepsilon_{\alpha i} \langle (\partial_i \Psi)(2\partial_\beta \nabla_\beta + \epsilon \nabla^2) \nabla_\alpha \psi \rangle = \epsilon^4 \nu \nabla^2 \nabla^2 \langle \psi \rangle. \tag{A 19}$$

The solvability conditions corresponding to various orders in ϵ are then obtained by inserting (A 3) into (A 19). It is easily checked that the solvability condition at order ϵ^n involves only $\psi^{(0)}, \psi^{(1)}, \ldots, \psi^{(n-2)}$.

The first non-trivial condition appears at order ϵ^3, namely

$$\varepsilon_{\alpha i} \langle (\partial_\beta Q_\gamma)(\partial_i \Psi) \rangle + \text{permutations of } \alpha, \beta, \gamma = 0, \quad \forall \alpha, \beta, \gamma. \tag{A 20}$$

The symmetrization in (α, β, γ) is needed, because this tensor is contracted with $\nabla_\alpha \nabla_\beta \nabla_\gamma \psi^{(0)}$. If we now assume that the basic flow Ψ is *parity-invariant*, i.e. Ψ is invariant under $\boldsymbol{x} \rightarrow -\boldsymbol{x}$, then we find from (A 12) that Q changes into $-Q$, and thus (A 20) is automatically satisfied.

At order ϵ^4, we obtain

$$\partial_T \nabla^2 \psi^{(0)} + 2\varepsilon_{\alpha i} \langle (\partial_i \Psi)(\partial_\beta \nabla_\beta \nabla_\alpha \psi^{(2)})\rangle + \varepsilon_{\alpha i} \langle (\partial_i \Psi)(\nabla^2 \nabla_\alpha \psi^{(1)})\rangle = \nu \nabla^2 \nabla^2 \psi^{(0)}. \quad \text{(A 21)}$$

Using (A 11) and (A 13) in (A 21) and observing that $\langle \psi^{(1)} \rangle$ and $\langle \psi^{(2)} \rangle$ do not contribute, we obtain the following closed equations for the (leading-order) large-scale perturbation $\psi^{(0)}$:

$$\partial_T \nabla^2 \psi^{(0)} = \nu_{\alpha\beta\gamma\eta} \nabla_\alpha \nabla_\beta \nabla_\gamma \nabla_\eta \psi^{(0)}. \quad \text{(A 22)}$$

Here, the eddy-viscosity tensor is given by

$$\nu_{\alpha\beta\gamma\eta} = \nu \delta_{\alpha\beta} \delta_{\gamma\eta} - \varepsilon_{\alpha i} \delta_{\beta\gamma} \langle Q_\eta \partial_i \Psi \rangle - 2\varepsilon_{\alpha i} \langle (\partial_\beta S_{\gamma\eta})(\partial_i \Psi)\rangle. \quad \text{(A 23)}$$

In the isotropic case (e.g. with three- or six-fold rotational invariance), the eddy-viscosity tensor reduces to a scalar ν_E and (A 22) reduces to

$$\partial_T \psi^{(0)} = \nu_E \nabla^2 \psi^{(0)}, \quad \text{(A 24)}$$

with

$$\nu_E = \nu - \langle Q_1 (\partial_2 \Psi)\rangle - 2\langle (\partial_1 S_{11})(\partial_2 \Psi)\rangle. \quad \text{(A 25)}$$

Appendix B. Multiscale technique (nonlinear)

In this Appendix we derive the various nonlinear large-scale equations discussed in §§8 and 9. Mirror-symmetric and chiral cases will be handled together, because the scalings in ϵ are the same. Since different scalings are needed for the cases of positive eddy viscosity and marginally negative eddy viscosity, these will be dealt with separately (in §§B 1 and B 2, respectively). Since the expansions in the nonlinear case are technically not very different from the linear case, already discussed in Appendix A, we shall omit some obvious details. The basic equation is the nonlinear Navier–Stokes equation for the perturbation ψ of the basic flow Ψ:

$$\partial_t \partial^2 \psi + J(\partial^2 \Psi, \psi) + J(\partial^2 \psi, \Psi) + J(\partial^2 \psi, \psi) = \nu \partial^2 \partial^2 \psi. \quad \text{(B 1)}$$

B.1. *Positive eddy-viscosity (nonlinear theory)*

As seen in §8.1, the leading-order term is $O(\epsilon^0)$, so we assume

$$\psi = \psi^{(0)} + \epsilon \psi^{(1)} + \epsilon^2 \psi^{(2)} + \dots. \quad \text{(B 2)}$$

The derivatives in (B 1) are again decomposed according to (A 2). The main difference with the linear case is the presence of the nonlinear term $J(\partial^2 \psi, \psi)$. To obtain the first three auxiliary equations (the equivalent of (A 4), (A 9) and (A 10)), it suffices to expand $J(\partial^2 \psi, \psi)$ up to second order in ϵ:

$$J(\partial^2 \psi, \psi) \rightsquigarrow \epsilon^2 [J(\partial^2 \psi^{(1)}, \psi^{(1)}) + \varepsilon_{\alpha\beta} (\partial_\alpha \partial^2 \psi^{(1)})(\nabla_\beta \psi^{(0)})] + O(\epsilon^3). \quad \text{(B 3)}$$

The first two levels of auxiliary equations (A 4) and (A 9) are unchanged. In particular, $\psi^{(1)}$ is still given by (A 11). The third level is modified by the extra nonlinearity. Thus, instead of (A 13), we now have

$$\psi^{(2)} = Y_{\alpha\beta} \nabla_\alpha \nabla_\beta \psi^{(0)} + Q \cdot \nabla \langle \psi^{(1)} \rangle + \langle \psi^{(2)} \rangle, \quad \text{(B 4)}$$

where

$$\tilde{A} Y_{\alpha\beta} = -\varepsilon_{ij} (\partial_i \partial^2 Q_\alpha)(\partial_j Q_\beta) - \varepsilon_{i\alpha} \partial_i \partial^2 Q_\beta. \quad \text{(B 5)}$$

As before, $Y_{\alpha\beta}$ depends only on fast derivatives. The solvability conditions are obtained as in the linear theory. In addition to (A 15)–(A 17), we must now use

$$\langle J(\partial^2\psi,\psi)\rangle \rightsquigarrow \epsilon^2\varepsilon_{\alpha\beta}\langle[\nabla_\alpha(2\partial\cdot\nabla+\epsilon\nabla^2)\psi][\partial_\beta\psi]\rangle + \epsilon^2\varepsilon_{\alpha\beta}\langle[\partial_\alpha(2\partial\cdot\nabla+\epsilon\nabla^2)\psi][\nabla_\beta\psi]\rangle + \epsilon^3\langle\mathscr{J}[(2\partial\cdot\nabla+\epsilon\nabla^2)\psi,\psi]\rangle. \quad \text{(B 6)}$$

Here, $\mathscr{J}$ is the Jacobian in slow (large-scales) variables. Consequently, the solvability condition (previously (A 19)) requires the right-hand side of (B 6) as an additional term on the left-hand side. The solvability condition at order ϵ^3 is unchanged when Ψ, the basic flow, is parity-invariant. The solvability condition at order ϵ^4 gives now a *nonlinear* equation for the (leading-order) large-scale perturbation $\psi^{(0)}$:

$$\partial_T\nabla^2\psi^{(0)} + 2\varepsilon_{\alpha i}\langle(\partial_i Q_\beta)(\partial_\gamma Q_\eta)\rangle\nabla_\alpha[(\nabla_\beta\psi^{(0)})(\nabla_\gamma\nabla_\eta\psi^{(0)})] + 2\varepsilon_{\alpha i}\langle(\partial_\beta Y_{\gamma\eta})(\partial_i\Psi)\rangle\nabla_\alpha\nabla_\beta[(\nabla_\gamma\psi^{(0)})(\nabla_\eta\psi^{(0)})] + \mathscr{J}(\nabla^2\psi^{(0)},\psi^{(0)}) = \nu_{\alpha\beta\gamma\eta}\nabla_\alpha\nabla_\beta\nabla_\gamma\nabla_\eta\psi^{(0)}. \quad \text{(B 7)}$$

Here $\nu_{\alpha\beta\gamma\eta}$ is the eddy-viscosity tensor given as before by (A 23).

We now specialize to the case of six-fold rotational invariance. All fourth-order tensors appearing in (B 7) are then isotropic, and the equation can be significantly simplified. Let us begin with the second term on the left-hand side of (B 7), which involves the tensor

$$\langle(\partial_i Q_\beta)(\partial_\gamma Q_\eta)\rangle. \quad \text{(B 8)}$$

Observe that

$$\langle(\partial_i Q_\beta)(\partial_\gamma Q_\eta)\rangle = -\langle(\partial_i\partial_\gamma Q_\beta)Q_\eta\rangle = -\langle Q_\beta(\partial_i\partial_\gamma Q_\eta)\rangle. \quad \text{(B 9)}$$

Thus, the tensor is symmetric in the pairs (i,γ) and (β,η). The most general fourth-order tensor with these symmetries and with six-fold rotational invariance is

$$\langle(\partial_i Q_\beta)(\partial_\gamma Q_\eta)\rangle = A_1\delta_{i\gamma}\delta_{\beta\eta} + A_2(\delta_{i\beta}\delta_{\gamma\eta}+\delta_{i\eta}\delta_{\gamma\beta}) + A_3(\varepsilon_{i\beta}\delta_{\gamma\eta}+\varepsilon_{i\eta}\delta_{\gamma\beta}+\varepsilon_{\gamma\beta}\delta_{i\eta}+\varepsilon_{\gamma\eta}\delta_{i\beta}), \quad \text{(B 10)}$$

where A_1, A_2 and A_3 are arbitrary real constants. Observe that the presence of the ε-terms is permitted as long as mirror symmetry is not assumed. The coefficients A_1, A_2 and A_3 may be expressed in terms of second-order moments of $\partial_\alpha Q_\beta$ by suitable choices of indices. We return to the second term on the left-hand side of (B 7), denoted Λ. Using (B 10) and expanding the slow derivative ∇_α which acts on two factors, we can write it as $\Lambda = \Lambda^{(1)} + \Lambda^{(2)}$ with

$$\Lambda^{(1)} = -4A_3(\nabla^2\psi^{(0)})(\nabla^2\psi^{(0)}), \quad \text{(B 11)}$$

$$\Lambda^{(2)} = 2A_2\mathscr{J}(\nabla^2\psi^{(0)},\psi^{(0)}) - 4A_3\nabla\psi^{(0)}\cdot\nabla\nabla^2\psi^{(0)}. \quad \text{(B 12)}$$

To establish (B 11) and (B 12), we have used the identity

$$\varepsilon_{\alpha\beta}\varepsilon_{\eta\gamma} = \delta_{\alpha\eta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\eta}. \quad \text{(B 13)}$$

It follows that:

$$\Lambda = 2A_2\mathscr{J}(\nabla^2\psi^{(0)},\psi^{(0)}) - 4A_3\nabla\cdot[(\nabla\psi^{(0)})(\nabla^2\psi^{(0)})]. \quad \text{(B 14)}$$

Now, we turn to the third term on the left-hand side of (B 7), denoted II. Similar manipulations give

$$\text{II} = 4B_2\mathscr{J}(\nabla^2\psi^{(0)},\psi^{(0)}) - 4(B_3+B_4)\nabla\cdot[(\nabla\psi^{(0)})(\nabla^2\psi^{(0)})], \quad \text{(B 15)}$$

where B_1, B_2, B_3, and B_4 are defined by

$$\langle(\partial_\beta Y_{\gamma\eta})(\partial_i\Psi)\rangle = B_1\delta_{i\beta}\delta_{\gamma\eta} + B_2(\delta_{i\gamma}\delta_{\beta\eta}+\delta_{i\eta}\delta_{\beta\gamma}) + B_3(\varepsilon_{i\gamma}\delta_{\beta\eta}+\varepsilon_{\beta\gamma}\delta_{i\eta}) + B_4(\varepsilon_{i\eta}\delta_{\beta\gamma}+\varepsilon_{\beta\eta}\delta_{i\gamma}), \quad \text{(B 16)}$$

which is again a consequence of the various symmetries. Collecting all terms, we finally obtain the following large-scale nonlinear equation:

$$\partial_T \nabla^2 \psi^{(0)} + a\mathscr{J}(\nabla^2\psi^{(0)}, \psi^{(0)}) + c\boldsymbol{\nabla}\cdot[(\boldsymbol{\nabla}\psi^{(0)})\nabla^2\psi^{(0)}] = \nu_E \nabla^2\nabla^2\psi^{(0)}, \tag{B 17}$$

where

$$a = 1 + 2A_2 + 4B_2, \tag{B 18}$$

$$c = -4A_3 - 4(B_3 + B_4). \tag{B 19}$$

Note that in the mirror-symmetric (non-chiral) case the coefficient c vanishes.

B.2. *Marginally negative eddy-viscosity* (*nonlinear theory*)

Let us denote by ν_c the value of the molecular viscosity ν at which ν_E vanishes for the first time when lowering ν. Generically, $(\partial \nu_E/\partial \nu)_{\nu=\nu_c} \neq 0$. If we now take $\nu = \nu_c(1-\epsilon^2)$, it follows that the eddy viscosity will be negative and $O(\epsilon^2)$. Hence, eddy viscosity terms in the large-scale equation will be of the same order as higher-order linear terms involving six space derivatives. A dominant-balance argument then indicates that spatial scales will be $O(\epsilon^{-1})$, temporal scales $O(\epsilon^{-4})$ and the stream function $O(\epsilon^2)$. Thus, we assume

$$\psi = \epsilon^2\psi^{(2)} + \epsilon^3\psi^{(3)} + \epsilon^4\psi^{(4)} + \dots, \tag{B 20}$$

and set $\boldsymbol{X} = \epsilon \boldsymbol{x}$ and $T = \epsilon^4 t$. Hence,

$$\partial_i \rightarrow \partial_i + \epsilon\nabla_i, \quad \partial_t \rightarrow \partial_t + \epsilon^4\partial_T. \tag{B 21}$$

Before engaging into full-fledged asymptotics, we observe that since ψ starts with $O(\epsilon^2)$ terms, the lowest-order equation is obviously

$$\mathrm{A}\psi^{(2)} = 0. \tag{B 22}$$

Thus, $\psi^{(2)}$ depends only on slow variables. So, below, all terms involving fast derivatives acting on $\psi^{(2)}$ will be omitted. It is easy to see that to obtain all relevant auxiliary equations, we must expand to order ϵ^6. For this, we use (B 20) and (B 21) to obtain:

$$\begin{aligned}\partial_t\partial^2\psi \rightsquigarrow{}& \epsilon^3\partial_t\partial^2\psi^{(3)} + \epsilon^4(\partial_t\partial^2\psi^{(4)} + 2\partial_t\partial\cdot\boldsymbol{\nabla}\psi^{(3)})\\ &+ \sum_{l=5}^{6}\epsilon^l(\partial_t\partial^2\psi^{(l)} + 2\partial_t\partial\cdot\boldsymbol{\nabla}\psi^{(l-1)} + \partial_t\nabla^2\psi^{(l-2)}) + O(\epsilon^7),\end{aligned} \tag{B 23}$$

$$\begin{aligned}J(\partial^2\psi, \Psi) \rightsquigarrow{}& \varepsilon_{\alpha\beta}(\partial_\beta\Psi)[\epsilon^3\partial_\alpha\partial^2\psi^{(3)}\\ &+ \epsilon^4(\partial_\alpha\partial^2\psi^{(4)} + 2\partial_\alpha\partial\cdot\boldsymbol{\nabla}\psi^{(3)} + \nabla_\alpha\partial^2\psi^{(3)})\\ &+ \sum_{l=5}^{6}\epsilon^l(\partial_\alpha\partial^2\psi^{(l)} + 2\partial_\alpha\partial\cdot\boldsymbol{\nabla}\psi^{(l-1)} + \nabla_\alpha\partial^2\psi^{(l-1)}\\ &+ \partial_\alpha\nabla^2\psi^{(l-2)} + 2\partial\cdot\boldsymbol{\nabla}\nabla_\alpha\psi^{(l-2)} + \nabla_\alpha\nabla^2\psi^{(l-3)})] + O(\epsilon^7),\end{aligned} \tag{B 24}$$

$$J(\partial^2\Psi, \psi) \rightsquigarrow \varepsilon_{\alpha\beta}(\partial_\alpha\partial^2\Psi)\sum_{l=3}^{6}\epsilon^l(\partial_\beta\psi^{(l)} + \nabla_\beta\psi^{(l-1)}) + O(\epsilon^7), \tag{B 25}$$

$$J(\partial^2\psi, \psi) \rightsquigarrow \epsilon^6[J(\partial^2\psi^{(3)}, \psi^{(3)}) + \varepsilon_{\alpha\beta}(\partial_\alpha\partial^2\psi^{(3)})(\nabla_\beta\psi^{(2)})] + O(\epsilon^7), \tag{B 26}$$

$$\begin{aligned}\nu_c(1-\epsilon^2)\partial^2\partial^2\psi \rightsquigarrow{}& \nu_c[\epsilon^3\partial^2\partial^2\psi^{(3)} + \epsilon^4(\partial^2\partial^2\psi^{(4)} + 4\partial\cdot\boldsymbol{\nabla}\partial^2\psi^{(3)})\\ &+ \epsilon^5(\partial^2\partial^2\psi^{(5)} + 4\partial\cdot\boldsymbol{\nabla}\partial^2\psi^{(4)} + 4\partial\cdot\boldsymbol{\nabla}\partial\cdot\boldsymbol{\nabla}\psi^{(3)}\\ &+ 2\partial^2\nabla^2\psi^{(3)} - \partial^2\partial^2\psi^{(3)})\\ &+ \epsilon^6(\partial^2\partial^2\psi^{(6)} + 4\partial\cdot\boldsymbol{\nabla}\partial^2\psi^{(5)} + 4\partial\cdot\boldsymbol{\nabla}\partial\cdot\boldsymbol{\nabla}\psi^{(4)}\\ &+ 2\partial^2\nabla^2\psi^{(4)} + 4\partial\cdot\boldsymbol{\nabla}\nabla^2\psi^{(3)} + \nabla^2\nabla^2\psi^{(2)}\\ &- \partial^2\partial^2\psi^{(4)} - 4\partial\cdot\boldsymbol{\nabla}\partial^2\psi^{(3)})] + O(\epsilon^7).\end{aligned} \tag{B 27}$$

Collecting in (B 23)–(B 27) terms $O(\epsilon^3)$, $O(\epsilon^4)$, $O(\epsilon^5)$ and $O(\epsilon^6)$, we obtain four equations of the form $\mathrm{A}f = g$ which allow us to determine $\psi^{(3)}$, $\psi^{(4)}$, $\psi^{(5)}$ and $\psi^{(6)}$. The solvability conditions attached to the $O(\epsilon^3)$ and $O(\epsilon^4)$ equations are automatically satisfied, since their right-hand sides are 'decorated' with fast derivatives everywhere on the left. We now write explicitly $\psi^{(3)}$, $\psi^{(4)}$, $\psi^{(5)}$ and $\psi^{(6)}$. At orders ϵ^3 and ϵ^4 we obtain, respectively

$$\psi^{(3)} = \langle\psi^{(3)}\rangle + \boldsymbol{Q}\cdot\boldsymbol{\nabla}\psi^{(2)}, \tag{B 28}$$

$$\psi^{(4)} = \langle\psi^{(4)}\rangle + \boldsymbol{Q}\cdot\boldsymbol{\nabla}\langle\psi^{(3)}\rangle + S_{\alpha\beta}\nabla_\alpha\nabla_\beta\psi^{(2)}. \tag{B 29}$$

Here, $\boldsymbol{Q} = (Q_\alpha)$ and $(S_{\alpha\beta})$ are the same quantities as in the linear theory, solutions of (A 12) and (A 14), respectively, but with $\nu = \nu_c$. At order ϵ^5 we have

$$\psi^{(5)} = \langle\psi^{(5)}\rangle + \boldsymbol{Q}\cdot\boldsymbol{\nabla}\langle\psi^{(4)}\rangle + S_{\alpha\beta}\nabla_\alpha\nabla_\beta\langle\psi^{(3)}\rangle + \tau_{\alpha\beta\gamma}\nabla_\alpha\nabla_\beta\nabla_\gamma\psi^{(2)} + \boldsymbol{P}\cdot\boldsymbol{\nabla}\psi^{(2)}. \tag{B 30}$$

The vector $\boldsymbol{P}$ and the tensor $(\tau_{\alpha\beta\gamma})$ depend only on fast variables and are the solutions of

$$\tilde{\mathrm{A}}\boldsymbol{P} = -\nu_c\partial^2\partial^2\boldsymbol{Q}, \tag{B 31}$$

$$\begin{aligned}\tilde{\mathrm{A}}\tau_{\alpha\beta\gamma} = {} & -2\partial_t\partial_\alpha S_{\beta\gamma} - \partial_t\delta_{\alpha\beta}Q_\gamma + \varepsilon_{\alpha i}(\partial^2\partial_i\Psi)S_{\beta\gamma} \\ & -2\varepsilon_{ij}(\partial_i\partial_\alpha S_{\beta\gamma})(\partial_j\Psi) - \varepsilon_{ij}\delta_{\alpha\beta}(\partial_i Q_\gamma)(\partial_j\Psi) \\ & -\varepsilon_{\alpha i}(\partial^2 S_{\beta\gamma})(\partial_i\Psi) - 2\varepsilon_{\alpha i}(\partial_\beta Q_\gamma)(\partial_i\Psi) - \varepsilon_{\alpha i}(\partial_i\Psi)\delta_{\beta\gamma} \\ & +4\nu_c\partial_\alpha\partial^2 S_{\beta\gamma} + 4\nu_c\partial_\alpha\partial_\beta Q_\gamma + 2\nu_c\delta_{\alpha\beta}\partial^2 Q_\gamma.\end{aligned} \tag{B 32}$$

At order ϵ^6 we obtain

$$\begin{aligned}\psi^{(6)} = {} & \langle\psi^{(6)}\rangle + \boldsymbol{Q}\cdot\boldsymbol{\nabla}\langle\psi^{(5)}\rangle + S_{\alpha\beta}\nabla_\alpha\nabla_\beta\langle\psi^{(4)}\rangle \\ & +\tau_{\alpha\beta\gamma}\nabla_\alpha\nabla_\beta\nabla_\gamma\langle\psi^{(3)}\rangle + \boldsymbol{P}\cdot\boldsymbol{\nabla}\langle\psi^{(3)}\rangle \\ & +Z_{\alpha\beta\gamma\eta}\nabla_\alpha\nabla_\beta\nabla_\gamma\nabla_\eta\psi^{(2)} + W_{\alpha\beta}\nabla_\alpha\nabla_\beta\psi^{(2)} + Y_{\alpha\beta}(\nabla_\alpha\psi^{(2)})(\nabla_\beta\psi^{(2)}).\end{aligned} \tag{B 33}$$

The tensor $Y_{\alpha\beta}$ is the same as in §B.1, solution of (B 5) with $\nu = \nu_c$. The tensors $Z_{\alpha\beta\gamma\eta}$ and $W_{\alpha\beta}$ are the zero-mean-value solutions of

$$\begin{aligned}\tilde{\mathrm{A}}Z_{\alpha\beta\gamma\eta} = {} & \varepsilon_{\alpha i}(\partial_i\partial^2\Psi)\tau_{\beta\gamma\eta} - 2\varepsilon_{ij}(\partial_i\partial_\alpha\tau_{\beta\gamma\eta})(\partial_j\Psi) \\ & -\varepsilon_{ij}(\partial_i S_{\alpha\beta})(\partial_j\Psi)\delta_{\gamma\eta} - \epsilon_{\alpha i}(\partial^2\tau_{\beta\gamma\eta})(\partial_i\Psi) \\ & -2\varepsilon_{\alpha i}(\partial_\beta S_{\gamma\eta})(\partial_i\Psi) - \varepsilon_{\alpha i}Q_\beta(\partial_i\Psi)\delta_{\gamma\eta} \\ & +4\nu_c\partial_\alpha\partial^2\tau_{\beta\gamma\eta} + 4\nu_c\partial_\alpha\partial_\beta S_{\gamma\eta} + 2\nu_c\partial^2 S_{\alpha\beta}\delta_{\gamma\eta} \\ & +4\nu_c\partial_\alpha Q_\beta\delta_{\gamma\eta} + \nu_c\delta_{\alpha\beta}\delta_{\gamma\eta},\end{aligned} \tag{B 34}$$

and

$$\begin{aligned}\tilde{\mathrm{A}}W_{\alpha\beta} = {} & \varepsilon_{\alpha i}(\partial_i\partial^2\Psi)P_\beta - 2\varepsilon_{ij}(\partial_i\partial_\alpha P_\beta)(\partial_j\Psi) \\ & -\varepsilon_{\alpha i}(\partial^2 P_\beta)(\partial_i\Psi) + 4\nu_c\partial_\alpha\partial^2 P_\beta - \nu_c\partial^2\partial^2 S_{\alpha\beta} - 4\nu_c\partial_\alpha\partial^2 Q_\beta.\end{aligned} \tag{B 35}$$

As for the solvability conditions, the intermediate equation playing now the role of (A 19) is

$$\begin{aligned}\epsilon^6\partial_T\nabla^2\langle\psi\rangle & + \epsilon^2\varepsilon_{\alpha\beta}\langle[\nabla_\alpha(2\partial\cdot\boldsymbol{\nabla} + \epsilon\nabla^2)\psi][\partial_\beta\Psi]\rangle \\ & + \epsilon^2\varepsilon_{\alpha\beta}\langle[\nabla_\alpha(2\partial\cdot\boldsymbol{\nabla} + \epsilon\nabla^2)\psi][\partial_\beta\psi]\rangle \\ & + \epsilon^2\varepsilon_{\alpha\beta}\langle[\partial_\alpha(2\partial\cdot\boldsymbol{\nabla} + \epsilon\nabla^2)\psi][\nabla_\beta\psi]\rangle \\ & + \epsilon^3\langle\mathcal{J}[(2\partial\cdot\boldsymbol{\nabla} + \epsilon\nabla^2)\psi, \psi]\rangle - \nu_c(1-\epsilon^2)\epsilon^4\nabla^2\nabla^2\langle\psi\rangle = 0.\end{aligned} \tag{B 36}$$

As before, $\mathcal{J}$ denotes the Jacobian in slow variables. The first non-trivial solvability conditions appears now at order ϵ^5, and is satisfied with parity-invariance. At order ϵ^6

the solvability condition is precisely the vanishing of the eddy viscosity for $\nu = \nu_c$. The solvability condition at order ϵ^7 is

$$\varepsilon_{\alpha\beta}\langle[\nabla_\alpha(2\partial\cdot\nabla\psi^{(5)}+\nabla^2\psi^{(4)})][\partial_\beta\Psi]\rangle = 0, \tag{B 37}$$

which is again satisfied with parity-invariance. Finally, at order ϵ^8 we obtain a closed nonlinear equation for $\psi^{(2)}$, namely

$$\begin{aligned}\partial_T\nabla^2\psi^{(2)} &+ 2\varepsilon_{\alpha i}\langle(\partial_i Q_\beta)(\partial_\gamma Q_\eta)\rangle\nabla_\alpha[(\nabla_\beta\psi^{(2)})(\nabla_\gamma\nabla_\eta\psi^{(2)})]\\ &+2\varepsilon_{\alpha i}\langle(\partial_\beta Y_{\gamma\eta})(\partial_i\Psi)\rangle\nabla_\alpha\nabla_\beta[(\nabla_\gamma\psi^{(2)})(\nabla_\eta\psi^{(2)})]+\mathscr{J}(\nabla^2\psi^{(2)},\psi^{(2)})\\ &= \Upsilon_{\alpha\beta\gamma\eta}\nabla_\alpha\nabla_\beta\nabla_\gamma\nabla_\eta\psi^{(2)}+T_{\alpha\beta\gamma\eta\xi\theta}\nabla_\alpha\nabla_\beta\nabla_\gamma\nabla_\eta\nabla_\xi\nabla_\theta\psi^{(2)}.\end{aligned} \tag{B 38}$$

Here

$$\Upsilon_{\alpha\beta\gamma\eta} = -2\varepsilon_{\alpha i}\langle(\partial_\beta W_{\gamma\eta})(\partial_i\Psi)\rangle-\delta_{\alpha\beta}\varepsilon_{\gamma i}\langle P_\eta\partial_i\Psi\rangle-\nu_c\delta_{\alpha\beta}\delta_{\gamma\eta}, \tag{B 39}$$

$$T_{\alpha\beta\gamma\eta\xi\theta} = -2\varepsilon_{\alpha i}\langle(\partial_\beta Z_{\gamma\eta\xi\theta})(\partial_i\Psi)\rangle-\varepsilon_{\alpha i}\delta_{\beta\gamma}\langle\tau_{\eta\xi\theta}\partial_i\Psi\rangle. \tag{B 40}$$

Note that (B 38) has the same nonlinearities as (B 7).

We now assume six-fold rotational invariance. All fourth-order tensors are then isotropic, which implies that the nonlinear terms in (B 38) are just those of (B 17). As for the tensor $T_{\alpha\beta\gamma\eta\xi\theta}$, since it is contracted with six ∇, we may replace it by its completely symmetrized form (one over 6! the sum over all the permutations of indices). Unfortunately, completely symmetric tensors of sixth order with six-fold rotational symmetry are not, in general, isotropic, but contain a two-parameter family of anisotropic tensors.

When isotropy of all relevant tensors holds, (B 38) reduces to

$$\partial_T\nabla^2\psi^{(0)}+a\mathscr{J}(\nabla^2\psi^{(0)},\psi^{(0)})+c\nabla\cdot[(\nabla\psi^{(0)})\nabla^2\psi^{(0)}] = -\mu_4\nabla^2\nabla^2\psi^{(0)}-\mu_6\nabla^2\nabla^2\nabla^2\psi^{(0)}. \tag{B 41}$$

Here, a and c are given, respectively, by (B 18) and (B 19), and

$$\mu_4 = 2\langle(\partial_1 W_{11})(\partial_2\Psi)\rangle+\langle P_1\partial_2\Psi\rangle+\nu_c, \tag{B 42}$$

$$\mu_6 = 2\langle(\partial_1 Z_{1111})(\partial_2\Psi)\rangle+\langle\tau_{111}\partial_2\Psi\rangle. \tag{B 43}$$

Appendix C. Low-order Reynolds-number expansion of the eddy viscosity

We give here the expressions for the first three non-vanishing coefficients in the expression (31) of the eddy viscosity ν_E in the isotropic case. They are derived by successive applications of the recursion relations (29), as explained in §4.1. The assumed isotropy enables us to simplify the final expressions. The intermediate algebra is not particularly enlightening and will be omitted. The result is (x and y have been used instead of x_1 and x_2 for brevity)

$$\nu_E = \nu+\nu^{(1)}\nu^{-1}+\nu^{(2)}\nu^{-2}+O(\nu^{-3}), \tag{C 1}$$

$$\nu^{(1)} = -\tfrac{1}{2}\langle(\partial_x\Psi)\partial^2(\partial_x\Psi)\rangle-\tfrac{1}{2}\langle(\partial_y\Psi)\partial^2(\partial_y\Psi)\rangle, \tag{C 2}$$

$$\begin{aligned}\nu^{(2)} = {}&\langle(\partial_y\Psi)\partial^{-2}\partial^{-2}[J(\partial_y\Psi,\Psi)+J(\partial^2\Psi,\partial^{-2}\partial_y\Psi)]\rangle\\ &+4\langle(\partial_{xy}\Psi)\partial^{-2}\partial^{-2}J(\Psi,\partial^{-2}\partial_{xy}\Psi)\rangle\\ &+2\langle(\partial_{xy}\Psi)\partial^{-2}\partial^{-2}[(\partial_y\Psi)(\partial_y\Psi)]\rangle\\ &-2\langle(\partial_{xy}\Psi)\partial^{-2}\partial^{-2}[(\partial^2\partial_y\Psi)(\partial^{-2}\partial_y\Psi)]\rangle\\ &-8\langle(\partial_{xxy}\Psi)\partial^{-2}\partial^{-2}\partial^{-2}[J(\Psi,\partial_y\Psi)+J(\partial^{-2}\partial_y\Psi,\partial^2\Psi)]\rangle\\ &+8\langle(\partial_{xy}\Psi)\partial^{-2}\partial^{-2}J(\partial^{-2}\partial^{-2}\partial_{xy}\Psi,\partial^2\Psi)\rangle.\end{aligned} \tag{C 3}$$

Note that (C 2) is already in DF.

Appendix D. Flow δ-correlated in time

As explained in §7.1, in the case of white noise, it is convenient to introduce a family of basic flows $\Psi_\eta(x,t)$ depending on a small parameter η. The flows Ψ_η are defined as

$$\Psi_\eta(x,t) \equiv \frac{1}{\eta}\Psi\left(x,\frac{t}{\eta^2}\right), \tag{D 1}$$

where $\Psi(x,t)$ is a random function, homogeneous stationary and time-reversal-symmetric with correlations decreasing sufficiently fast at infinity. Basic flows δ-correlated in time are then obtained as limits of flows with short correlation time.

We have verified that mostly the same formalism used in the case of deterministic, space–time periodic flows remains applicable in the random case with the following modifications: (i) since the basic flow depends on η, then Q, $S_{\alpha\beta}$ appearing in the eddy-viscosity (22) and the latter itself will be functions of η (the dependence is omitted for conciseness), and have well-defined limits for $\eta\to 0$; (ii) averages $\langle\cdot\rangle$ must be interpreted as ensemble averages; (iii) the solutions of auxiliary problems must be in a suitable class of functions (see below).

The auxiliary problems to be solved, already given as (23) and (24), are

$$\mathrm{A}Q_\alpha = \epsilon_{\alpha i}\partial_i\partial^2\Psi_\eta, \tag{D 2}$$

$$\begin{aligned}\mathrm{A}S_{\alpha\beta} = &-2\partial_t\partial_\alpha Q_\beta - \epsilon_{i\alpha}(\partial_i\partial^2\Psi_\eta)Q_\beta \\ &+2\epsilon_{ij}(\partial_i\Psi_\eta)(\partial_j\partial_\alpha Q_\beta)+\epsilon_{i\alpha}(\partial_i\Psi_\eta)(\partial^2 Q_\beta)+4\nu\partial_\alpha\partial^2 Q_\beta,\end{aligned} \tag{D 3}$$

where
$$\mathrm{A} = \partial_t\partial^2 + J(\partial^2\bullet,\Psi_\eta)+J(\partial^2\Psi_\eta,\bullet)-\nu\partial^2\partial^2. \tag{D 4}$$

We now take advantage of the presence of the small parameter η. We introduce the 'very fast' time variable $\tau = t/\eta^2$, which is even faster than the fast time t. We require the solutions of (D 2) and (D 3) to be homogeneous with respect to the space variables, stationary with respect to t and asymptotically first-order stationary with respect to τ, i.e. the average of the solutions become independent of τ, as $\tau\to+\infty$.

In the two-time formalism involving t and τ, the first auxiliary equation becomes

$$\left(\partial_t+\frac{1}{\eta^2}\partial_\tau\right)\partial^2 Q_\alpha+\frac{1}{\eta}J(\partial^2 Q_\alpha,\Psi)+\frac{1}{\eta}J(\partial^2\Psi,Q_\alpha) = \nu\partial^2\partial^2 Q_\alpha+\frac{1}{\eta}\epsilon_{\alpha i}\partial_i\partial^2\Psi, \tag{D 5}$$

where $\Psi = \Psi(x,\tau)$. The solutions $Q = Q(x,\tau,t)$ is sought as a series in η: $Q = Q^{(0)}+\eta Q^{(1)}+\eta^2 Q^{(2)}+\dots$, where the upper index now indicates the order in η. Order-$1/\eta^2$ contributions from (D 5) give

$$\partial_\tau\partial^2 Q_\alpha^{(0)} = 0. \tag{D 6}$$

Hence, $Q^{(0)}$ is independent of the very fast time τ. Order-$1/\eta$ contributions from (D 5) give

$$\partial_\tau\partial^2 Q_\alpha^{(1)} = \epsilon_{\alpha i}\partial_i\partial^2\Psi. \tag{D 7}$$

Hence
$$Q_\alpha^{(1)} = \epsilon_{\alpha i}\partial_i\int_0^\tau \Psi(x,s)\,\mathrm{d}s. \tag{D 8}$$

It is easily checked that adding a τ-independent term of zero mean value and dependent on the history of Ψ prior to $t=0$ makes no difference in what follows.

Similarly, for the quantities $S_{\alpha\beta}$, we find that $S^{(0)}_{\alpha\beta}$ is independent of τ and

$$S^{(1)}_{\alpha\beta} = -2\epsilon_{\alpha i}\,\partial^{-2}\,\partial_i\,\partial_\beta \int_0^\tau \Psi(\boldsymbol{x}, s)\,\mathrm{d}s. \tag{D 9}$$

Let us now consider all the non-trivial solvability conditions encountered in the multiscale expansion. As in the deterministic case, the equation expressing the absence of an AKA-effect (solvability of third order in ϵ) is

$$(1/\eta)\,\epsilon_{\alpha i}\langle(\partial_\beta Q_\gamma)(\partial_i \Psi)\rangle + \text{permutations of } \alpha, \beta, \gamma = 0, \quad \forall \alpha, \beta, \gamma. \tag{D 10}$$

In the main body of the paper, we have to assume parity-invariance to ensure the vanishing of this term. With white-noise time-dependence, (D 10) is automatically satisfied. Since $\eta \rightarrow 0$, it suffices to check the vanishing of the $O(1/\eta)$ and the $O(\eta^0)$ contributions from (D 10). The former vanishes because $Q^{(0)}$ is independent of the very fast time and $\langle \Psi \rangle = 0$. As for the latter, it involves

$$R_{\alpha\beta\gamma} = \varepsilon_{\alpha i}\langle(\partial_\beta Q^{(1)}_\gamma)(\partial_i \Psi)\rangle. \tag{D 11}$$

In view of (D 9), we obtain

$$R_{\alpha\beta\gamma} = \varepsilon_{\alpha i}\,\varepsilon_{\gamma j}\int_0^{+\infty} \langle(\partial_\beta \partial_j \Psi(\boldsymbol{x}, s))(\partial_i \Psi(\boldsymbol{x}, 0))\rangle\,\mathrm{d}s. \tag{D 12}$$

By stationarity and time-reversal symmetry

$$R_{\alpha\beta\gamma} = \tfrac{1}{2}\varepsilon_{\alpha i}\,\varepsilon_{\gamma j}\int_{-\infty}^{+\infty} \langle(\partial_\beta \partial_j \Psi(\boldsymbol{x}, s))(\partial_i \Psi(\boldsymbol{x}, 0))\rangle\,\mathrm{d}s. \tag{D 13}$$

By homogeneity,

$$R_{\alpha\beta\gamma} = -\tfrac{1}{2}\varepsilon_{\alpha i}\,\epsilon_{\gamma j}\int_{-\infty}^{+\infty} \langle(\partial_i \Psi(\boldsymbol{x}, s))(\partial_\beta \partial_j \Psi(\boldsymbol{x}, 0))\rangle\,\mathrm{d}s. \tag{D 14}$$

Shifting the time arguments by $-s$ and redefining $-s$ as s, we obtain

$$R_{\alpha\beta\gamma} = -\tfrac{1}{2}\varepsilon_{\alpha i}\,\varepsilon_{\gamma j}\int_{-\infty}^{+\infty} \langle(\partial_i \Psi(\boldsymbol{x}, 0))(\partial_\beta \partial_j \Psi(\boldsymbol{x}, s))\rangle\,\mathrm{d}s. \tag{D 15}$$

Comparison of (D 13) and (D 15) shows that $R_{\alpha\beta\gamma} = 0$. Thus, the solvability condition (D 10) is satisfied. What we have shown is actually the absence of the AKA-effect for δ-correlated flows, irrespective of parity-invariance (Frisch *et al.* 1987*b*).

The next non-trivial solvability condition is the one that gives the eddy viscosity (21), which in the notation of this Appendix reads

$$\partial_T \nabla^2 \langle \psi^{(0)} \rangle = \nu_{\alpha\beta\gamma\zeta} \nabla_\alpha \nabla_\beta \nabla_\gamma \nabla_\zeta \langle \psi^{(0)} \rangle, \tag{D 16}$$

with $$\nu_{\alpha\beta\gamma\zeta} = \nu\delta_{\alpha\beta}\delta_{\gamma\zeta} - (1/\eta)\,\epsilon_{\alpha i}\,\delta_{\beta\gamma}\langle Q_\zeta \partial_i \Psi\rangle - (2/\eta)\,\epsilon_{\alpha i}\langle(\partial_\beta S_{\gamma\zeta})(\partial_i \Psi)\rangle. \tag{D 17}$$

It is now easily shown that the eddy-viscosity tensor has a finite limit, as $\eta \rightarrow 0$. The only 'dangerous' terms are those which might be $O(1/\eta)$, but they vanish because the leading contributions to Q_α and $S_{\alpha\beta}$ are independent of the very fast variable and because $\langle \Psi \rangle = 0$. The $O(\eta^0)$ terms give a finite eddy viscosity tensor:

$$\nu_{\alpha\beta\gamma\zeta} = \nu\delta_{\alpha\beta}\delta_{\gamma\zeta} + \tfrac{1}{2}\delta_{\alpha\beta}\int_{-\infty}^{+\infty} \langle \Psi(\boldsymbol{x}, s)\,(\partial^2\delta_{\gamma\zeta} - \partial_\gamma \partial_\zeta)\,\Psi(\boldsymbol{x}, 0)\rangle\,\mathrm{d}s$$
$$+ 2\int_{-\infty}^{+\infty} \langle \Psi(\boldsymbol{x}, s)\,\partial_\alpha \partial_\beta(\partial_\gamma \partial_\zeta \partial^{-2} - \delta_{\gamma\zeta})\,\Psi(\boldsymbol{x}, 0)\rangle\,\mathrm{d}s. \tag{D 18}$$

The terms expressing the effect of the small-scale flow give two contributions with opposite signs. If the small-scale flow is sufficiently anisotropic, the eddy viscosity tensor can have negative eigenvalues, corresponding to directions of amplification of the large-scale perturbation. However, in the isotropic case, it is easily shown that the two contributions cancel. As a consequence, the eddy viscosity is exactly equal to the molecular viscosity.

Essentially the same method as used in this Appendix can be applied to calculate the eddy viscosity of a D-dimensional small-scale velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ which is δ-correlated in time. The expression of the eddy viscosity tensor appearing in the equations for the large-scale velocity field is

$$\nu_{\alpha\beta\gamma\zeta} = \nu\delta_{\alpha\beta}\delta_{\gamma\zeta}+\tfrac{1}{2}\delta_{\alpha\zeta}\int_{-\infty}^{+\infty}\langle v_\beta(\boldsymbol{x},s)\,v_\gamma(\boldsymbol{x},0)\rangle\,\mathrm{d}s$$
$$-\int_{-\infty}^{+\infty}\langle v_\beta(\boldsymbol{x},s)\,\partial_\alpha\partial_\zeta\partial^{-2}v_\gamma(\boldsymbol{x},0)\rangle\,\mathrm{d}s-\int_{-\infty}^{+\infty}\langle v_\alpha(\boldsymbol{x},s)\,\partial_\beta\partial_\zeta\partial^{-2}v_\gamma(\boldsymbol{x},0)\rangle\,\mathrm{d}s. \quad \text{(D 19)}$$

In the isotropic case, this expression reduces to

$$\nu_E = \nu+\frac{2+D-D^2}{2-D-D^2}\frac{F}{2}, \quad \text{(D 20)}$$

where F is the constant appearing in the correlation function

$$\langle \boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{v}(\boldsymbol{x},0)\rangle = F\delta(t). \quad \text{(D 21)}$$

Note that the dependence of ν_E on F can be captured by an argument *à la* Maxwell, of the sort used in kinetic theory to derive the expression of the viscosity in a model where the particles are scattered elastically and isotropically. The D-dependent factor, which tends monotonically to one for D tending to infinity, can be traced back to incompressibility. In $D=2$, we recover the result $\nu_E=\nu$. For any dimension $D>2$, the contribution of the small-scale flow to the eddy viscosity is positive in the isotropic case.

Appendix E. Flow with symmetric streamlines

In this Appendix, we shall consider static (time-independent) basic flow leading to an isotropic eddy-viscosity tensor and having mirror-symmetric streamlines. The Navier–Stokes equation being rotation-invariant, without loss of generality, we may assume that

$$\Psi(-x,y) = \Psi(x,y). \quad \text{(E 1)}$$

We shall show that ν_E is an odd function of ν. Hence, the expansion of ν_E/ν in powers of $1/\nu$ will contain only even powers.

In order to calculate the eddy viscosity in the isotropic case, it is enough to consider the component ν_{1111}, which involves the 'fast' functions Q_1 and S_{11} introduced in §4. The equation for Q_1 is

$$\tilde{\mathrm{A}}Q_1 = \partial_y\partial^2\Psi. \quad \text{(E 2)}$$

This equation is invariant under the operations of mirror symmetry and simultaneous changes of sign of the viscosity and of Q_1. Specifically, the equation is invariant under

$$x\mapsto -x, \quad y\mapsto y, \quad Q_1\mapsto -Q_1, \quad \nu\mapsto -\nu. \quad \text{(E 3)}$$

Note that changing the sign of the viscosity would be a bad idea for the time-dependent problem, but is quite harmless in the time-independent case.

Let us now consider the equation for S_{11}:

$$\tilde{A}S_{11} = -2(\partial_y \Psi)\,\partial_{xx} Q_1 + 2(\partial_x \Psi)\,\partial_{xy} Q_1 - (\partial_y \Psi)\,\partial^2 Q_1 + (\partial_y \partial^2 \Psi)\,Q_1 + 4\nu\,\partial_x \partial^2 Q_1. \quad \text{(E 4)}$$

Using the above symmetry properties of Q_1, it is easy to verify that the equation is invariant under

$$x \mapsto -x, \quad y \mapsto y, \quad S_{11} \mapsto S_{11}, \quad \nu \mapsto -\nu. \quad \text{(E 5)}$$

From (A 25), we have

$$\frac{\nu_E}{\nu} = 1 - \frac{1}{\nu}\langle Q_1 \partial_y \Psi\rangle - \frac{2}{\nu}\langle(\partial_x S_{11})(\partial_y \Psi)\rangle. \quad \text{(E 6)}$$

It follows from the symmetries of the functions Q_1 and S_{11}, that the right-hand side of (E 6) is an even function of ν. This completes the proof.

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